The torsion in the cohomology of wild elliptic fibers

Inaugural-Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Heinrich-Heine-Universität Düsseldorf

vorgelegt von

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Düsseldorf, April 2019

aus dem Mathematischen Institut
der Heinrich-Heine-Universität Düsseldorf

Gedruckt mit der Genehmigung der
Mathematisch-Naturwissenschaftlichen Fakultät der
Heinrich-Heine-Universität Düsseldorf

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Korreferent: Prof. Dr. Immanuel Halupczok

Tag der mündlichen Prüfung: 28.06.2019

Zusammenfassung

Elliptische Faserungen spielen eine wichtige Rolle innerhalb der Klassifikation von algebraischen Flächen. Sie werden definiert als integere, reguläre Flächen mit einem surjektiven, eigentlichen Morphismus auf eine reguläre Kurve, so dass die generische Faser glatt und geometrisch integer ist von Geschlecht eins. Ist $f\colon X \to S$ eine elliptische Faserung, so lässt sich $R^1 f_* \mathscr{O}_X$ als Summe einer invertierbaren Garbe $\mathscr{L}$ und eines Torsionsmoduls $\mathscr{T}$ zerlegen. Letzterer ist nur an endlich vielen Punkten nicht-trivial. In Charakteristik Null verschwindet dieser sogar vollständig, doch in positiver Charakteristik ist dies i.A. nicht der Fall. Fasern über abgeschlossene Punkte $s \in S$ mit $\mathscr{T}_s$ nicht-trivial nennen wir *wilde Fasern*. In dieser Arbeit untersuchen wir die Struktur von $\mathscr{T}$.

Für eine elliptische Faserung $f\colon X \to S$ bezeichne $X_s = mF$ die Faser von X über dem abgeschlossenen Punkt s in S, wobei wir X_s als größtmögliches Vielfaches eines Divisors F auffassen. Bombieri und Mumford zeigten, dass sich m schreiben lässt als Produkt νp^e, wobei p die Charakteristik des Restekörpers sei und ν die Ordnung des Normalenbündels $\mathscr{O}_F(F)$ in $\mathrm{Pic}(F)$. Abhängig von der Entwicklung der Ordnung von $\mathscr{O}_{nF}(F)$ in $\mathrm{Pic}(nF)$ lässt sich das Wachstum von $\dim_k(H^1(nF, \mathscr{O}_{nF}))$ mithilfe der Arbeit von Bertapelle und Tong beschreiben. Aus diesem Wachstum schlussfolgern wir eine Formel für den Torsionsanteil von $R^1 f_* \mathscr{O}_X$.

Um eine besser handhabbare Formel zu erlangen, versuchen wir, eine elliptische Faserung $X \to S$ als Quotient einer anderen elliptischen Faserung zu gewinnen. Es genügt, dies im Falle des Spektrums eines vollständigen diskreten Bewertungsringes R zu tun. Sei dazu G die Galoisgruppe einer minimalen Körpererweiterung K' über K, so dass die generische Faser X_K einen K'-wertigen Punkt besitzt. Ferner notiere durch R' den ganzen Abschluss von R in K'. Wir gewinnen dann die elliptische Faserung in bestimmten Fällen als Quotient einer elliptischen Faserung $X' \to \mathrm{Spec}(R')$. Das Anwenden einer Spektralsequenz für Räume mit Gruppenwirkung zeigt, dass der Torsionsmodul isomorph zur Gruppenkohomologie $H^1(G, R')$ ist. Dessen Struktur wurde im wild verzweigten, zyklischen Fall bereits von Sen untersucht. Wir verallgemeinern unsere Methode in höheren Dimensionen.

Ferner wenden wir unsere Kenntnis der Torsionsstruktur in $R^1 f_* \mathscr{O}_X$ auf algebraische Flächen von Kodaira-Dimension $-\infty$ und Null an. Dadurch können wir insbesondere die Existenz zweier potentiell möglicher hyperelliptischen Flächen ausschließen.

Summary

Elliptic fibrations play an eminent role in the classification of algebraic surfaces. They are defined as regular integral surfaces together with a surjective, proper morphism to a regular curve, so that the generic fiber is smooth and geometrically integral of genus one. If $f\colon X \to S$ is an elliptic fibration, we can write $R^1 f_* \mathscr{O}_X$ as a sum of an invertible sheaf $\mathscr{L}$ and a torsion module $\mathscr{T}$. The latter one is only non-trivial at finitely many points. In characteristic zero, it even vanishes completely. Yet, in positive characteristic, this is in general not the case. Fibers over closed points $s \in S$ such that $\mathscr{T}_s$ is non-trivial are called *wild fibers*. We examine their torsion structure in this thesis.

For an elliptic fibration $f\colon X \to S$, we denote by $X_s = mF$ the fiber of X over the closed point s in S, where we consider X_s as the largest multiple of a divisor F. Bombieri and Mumford showed that m can be written as a product νp^e, where p is the characteristic of the residue field and ν the order of the normal bundle $\mathscr{O}_F(F)$ in $\mathrm{Pic}(F)$. Depending on the growth of the order of $\mathscr{O}_{nF}(F)$ in $\mathrm{Pic}(nF)$, one obtains a description of the growth of $\dim_k(H^1(nF, \mathscr{O}_{nF}))$ by the work of Bertapelle and Tong. We use these results to deduce a formula for the torsion part of $R^1 f_* \mathscr{O}_X$.

To obtain a more manageable formula, we try to construct an elliptic fibration $f\colon X \to S$ as a quotient of another elliptic fibration. It suffices to do this over a complete discrete valuation ring R. To do this, let G be the Galois group of a minimal field extension K' over K so that the generic fiber X_K admits a K'-valued point. Moreover, denote by R' the integral closure of R in K'. We will then construct the elliptic fibration as a quotient of an elliptic fibration $X' \to \mathrm{Spec}(R')$ in several cases. By applying a spectral sequence on spaces with a group action, we deduce that the torsion module is isomorphic to the group cohomology $H^1(G, R')$. Its structure in the wildly ramified cyclic case was already studied by Sen. We generalize our method to higher dimensions.

Furthermore, we apply our knowledge of the structure of the torsion part of $R^1 f_* \mathscr{O}_X$ on algebraic surfaces of Kodaira dimension $-\infty$ and zero. This enables us in particular to exclude the existence of two potentially possible hyperelliptic surfaces.

Contents

Introduction

Elliptic curves are objects that appear in many areas of mathematics. From an arithmetical point of view, they appear when studying the solutions of a cubic equation in two variables over a field K. If the equation is "nice", i.e. the corresponding curve is a smooth genus-one curve which admits at least one solution, we can manipulate the equation such that we obtain an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K,$$

also called *Weierstraß equation*. If the characteristic of K is different from two or three, one can even achieve a Weierstraß equation of the form $y^2 = x^3+a_4x+a_6$. The homogenization of the former one gives the corresponding smooth genus-one curve

$$C = V_+(Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3) \subset \mathbb{P}^2_K.$$

In fact, every smooth genus-one curve with a fixed rational point $\mathcal{O}$ can be given that way, where $\mathcal{O}$ corresponds to the point $(0 : 1 : 0) \in C$. Whereas the solutions of linear or quadratic equations in two variables are fairly well understood, e.g. by the Hasse–Minkowski Theorem for K a number field, it is very hard to find all points (if any except $(0 : 1 : 0)$) on elliptic curves. They present complexities which are still of interest in current research. For example, Wiles proof of Fermat's Last Theorem relies on the theory of elliptic curves.

Elliptic curves come with a commutative group structure which is determined by choosing a rational point as the neutral element. This turns an elliptic curve into an abelian variety. These are exactly the abelian varieties of dimension one. Altogether, elliptic curves are complex but accessible enough to provide a standard testing ground for proofs and techniques on curves of higher genus or general abelian varieties. This is also the case in this thesis, as we will see.

Now assume that the coefficients a_i of a Weierstraß equation of an elliptic curve E_K over K are all elements of a ring R with field of fractions K. One naturally asks if the properties of an elliptic curve carry over to the scheme $W \subset \mathbb{P}^2_R$ given by the Weierstraß equation, also called *Weierstraß model*. The answer is in general no, because fibers over points can degenerate to non-smooth curves like cusps and nodes. Moreover, the appearance of singular fibers means that W is not smooth over R, so in particular no abelian scheme. Often it is not even regular. This "failure" leads to the following two generalizations: The *Néron model* of an elliptic curve and an *elliptic fibration.* The former one generalizes the property of being an abelian variety: It asks for a smooth separated commutative group scheme N over R of finite type with generic fiber isomorphic to E_K, satisfying the *Néron mapping property*: For each smooth R-scheme Y and each K-morphism $u_K\colon Y_K \to N_K$, there is a unique R-morphism $u\colon Y \to N$ extending u_K. Néron models are of great interest, and we need results about the "degeneration" of its closed fibers, namely when the Néron model has semi-abelian reduction.

An elliptic fibration $f\colon X \to \mathrm{Spec}(R)$ is a proper morphism from an integral, regular scheme X to $\mathrm{Spec}(R)$ such that the canonical map $R \to f_*\mathscr{O}_X$ is an isomorphism and the generic fiber is a smooth geometrically integral genus-one curve. This definition generalizes the property of being a regular proper scheme. Note that the generic fiber is in general not an elliptic curve as it might lack a rational point. For an elliptic fibration $f\colon X \to \mathrm{Spec}(R)$, all but finitely many fibers are elliptic curves. When the generic fiber is an elliptic curve and R is an algebra over an algebraically closed field, then there are only finitely many possible singular fiber types classified by Kodaira and Néron, cf. Theorem 1.12. By a result of Liu, Lorenzini and Raynaud, this classification generalizes to every elliptic fibration, cf. Theorem 1.13.

Going a step further, we "globalize" the definition by taking a smooth projective integral curve S over a field k as the base, i.e. we say that $f\colon X \to S$ is an elliptic fibration if it satisfies the same properties mentioned in the definition.

Our starting point are elliptic fibrations over smooth projective curves defined over an algebraically closed field. They form an essential part of the Kodaira classification of algebraic surfaces over an algebraically closed field k. The Kodaira classification subdivides algebraic surfaces into four different classes according to the degree of the polynomial growth of its *plurigenera*: Given a regular integral surface X proper over k, we define its plurigenera as $P(n) = \dim_k(H^0(X, \omega_X^{\otimes n}))$. It grows like a polynomial of degree $d \leq 2$, and we set $\kappa(X) = d$ as the *Kodaira dimension of* X. Each surface X with $\kappa(X) = 1$ is an elliptic surface (or an quasi-elliptic surface, which only appear in characteristic two and three), so elliptic surfaces make up a whole class in the Kodaira classification. Moreover, they also appear in smaller Kodaira dimensions.

If S is a smooth proper curve over an algebraically closed field k and $f\colon X \to S$ is a *relatively minimal* elliptic fibration, i.e. there is no elliptic fibration $Y \to S$ such that $X \to S$ is obtained by blowing up a point in a closed fiber of Y, there is the famous *canonical bundle formula* for elliptic surfaces over $\mathbb{C}$ by Kodaira [37]. It was generalized to arbitrary characteristic by Bombieri and Mumford [7], stating that

$$\omega_X = f^*(\mathscr{L}^\vee \otimes \omega_S) \otimes \mathscr{O}_X\Big(\sum_{s \neq \eta} a_s F_s\Big)$$

holds with the following notation (cf. Theorem 1.9): $\mathscr{L}$ is an invertible sheaf on S and $F_s = m_s^{-1} X_s$ are divisors, where m_s is the greatest common divisor of multiplicities of the irreducible components appearing in the fiber X_s over a closed point $s \in S$. We denote by η the generic point of S and a_s is an integer between 0 and $m_s - 1$. If the generic fiber X_η has a rational point, so it is an elliptic curve, the multiplicity m_s equals one for every closed point. If X_η does not admit a rational point, this property again holds except for finitely many closed fibers. Now the difference to the complex case lies in the numbers a_s. If X is defined over $\mathbb{C}$, the equality $a_s = m_s - 1$ holds. In arbitrary characteristic, this depends on the structure of $R^1 f_* \mathscr{O}_X$: It decomposes into the sum of an invertible sheaf $\mathscr{L}$ on S and a torsion module $\mathscr{T}$ supported at finitely many closed points. In characteristic zero, the torsion part does not appear, cf. [38], Corollary 3.9. In positive characteristic, this can happen. Fibers X_s over closed points $s \in S$ such that s is in the support of $\mathscr{T}$ will be called *wild fibers*, whereas all other closed fibers will be called *tame*. For tame fibers, a_s is equal to $m_s - 1$ like in characteristic zero. In positive characteristic, a_s can take any value between 0 and $m_s - 1$. To study the structure of $\mathscr{T}$, a feature only appearing in positive characteristic, is the main purpose of this thesis.

This is done in two different ways. It is not difficult to see that we may assume the base S of an elliptic fibration $f\colon X \to S$ to be the spectrum of a complete discrete valuation ring, cf. Proposition 2.1. Let us write $X_s = mF$ for the single closed fiber as in the canonical bundle formula above. Restricting the invertible sheaf $\mathscr{O}_X(F)$ to the subscheme nF gives an element $\mathscr{O}_{nF}(F) \in \operatorname{Pic}(nF)$ of finite order. Bertapelle and Tong [5] describe the behaviour of the function $n \mapsto \operatorname{ord}(\mathscr{O}_{nF}(F))$ relying on unpublished work of Raynaud. It turns out that this function is monotonic increasing and bounded by $m = \nu p^e$, where $\nu = \operatorname{ord}(\mathscr{O}_F(F))$. Denoting by n_i the smallest positive integer such that $\operatorname{ord}(\mathscr{O}_{n_i F}(F))$ is equal to νp^i, we obtain positive integers α_i for $0 \leq i \leq e$ such that $n_i + a = \alpha_i \nu p^i$ holds (cf. Lemma 2.5). Note that a is the same one that appears in the canonical bundle formula. Moreover, we obtain a characterization of the growth of $H^1(nmF, \mathscr{O}_{nmF})$ which, together with the isomorphisms $R^1 f_* \mathscr{O}_X \otimes_R R/(\pi^n) \to H^1(nmF, \mathscr{O}_{nmF})$ coming from the theorem on semi-continuity and base change, yields the following description of the torsion in $R^1 f_* \mathscr{O}_X$: Writing $n_i = \beta_i m + \gamma_i$ with $0 \leq \gamma_i < m$ for $1 \leq i \leq 1$, we define

$$\begin{aligned} y_j &= (p-1)\big(\alpha_j - \beta_j p^{e-j} - \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor\big) - (\alpha_0 - 1)_{j-1}, \\ z_j &= (p-1)\Big((\beta_j + 1)p^{e-j} + \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor - \alpha_j\Big) + (\alpha_0 - 1)_{j-1} \end{aligned}$$

for $j = 1, \ldots, e$, where $\sum_{i \geq 0}(\alpha_0 - 1)_i p^i$ is the p-adic extension of $\alpha_0 - 1$. Then:

Theorem (see Theorem 2.11). *We have $R^1 f_* \mathscr{O}_X \simeq R \oplus \mathscr{T}$, with torsion part $\mathscr{T}$ given by*

$$\mathscr{T} = \bigoplus_{j=1}^{e} \left((R/\mathfrak{m}^{\beta_j + 1})^{\oplus y_j} \oplus (R/\mathfrak{m}^{\beta_j})^{\oplus z_j} \right).$$

Note that if the generic fiber X_K is an elliptic curve, we have $m_s = 1$. Hence, Theorem 2.11 tells us that $R^1 f_* \mathscr{O}_X$ has trivial torsion part. The theorem is proven by arranging skillfully terms, but the determination of the n_i is quite difficult. Moreover, relatively minimal elliptic fibrations over Dedekind schemes S are in one-to-one correspondence to smooth genus-one curves over the function field of S, and the theorem sheds no light on how the torsion can be expressed in terms of the generic fiber.

We therefore use a different, more geometric approach, which also works in higher dimension. This leads to the notion of an *abelian fibration* $X \to S$, which is essentially the same as the notion of elliptic fibration. The only difference lies in demanding that the generic fiber X_K is a smooth geometrically integral scheme which becomes isomorphic to an abelian variety after a field extension. The condition on the generic fiber X_K is equivalent to the statement that the generic fiber is a torsor under an abelian variety A_K, i.e. A_K acts on X_K such that after some field extension, there is some A_K-equivariant isomorphism between X_K and A_K (cf. Proposition 3.3). Again, to study the torsion structure, it suffices to work over a complete discrete valuation ring R. Assume that its residue field k is algebraically closed. Let X_K be a torsor under an abelian variety A_K which is the generic fiber of an abelian scheme $A \to \mathrm{Spec}(R)$. We take a minimal Galois extension K'/K with Galois group G so that $X_K(K')$ is non-empty. Let R' be the normalization of R in K' and $A' = A \otimes_R R'$. This is again an abelian scheme, on which G acts over the second factor. We twist this action by a cocycle corresponding to X_K, that is, by some K'-valued points on A', such that the quotient will give an abelian fibration $f \colon X \to \mathrm{Spec}(R)$ with generic fiber isomorphic to X_K. The upshot is that if the quotient morphism is étale, we can easily apply a spectral sequence for spaces with group actions on it to the quotient morphism that expresses the torsion in $H^1(X, \mathscr{O}_X)$ in terms of group cohomology.

Theorem (see Theorem 7.3). *In above situation, the torsion in $H^1(X, \mathscr{O}_X)$ is given by $H^1(G, R')$.*

In this situation, the extension K' over K is unique and its Galois group G is an abelian group which in the case of elliptic curves is even cyclic, cf. Proposition 7.2. Applying Sen's Theorem (cf. Theorem 5.21), $H^1(G, R')$ can be given explicitly in terms of ramification breaks of the higher ramification group filtration. When $X \to S$ is an elliptic fibration with X_η being a torsor under the elliptic curve A_K, we let $A \to S$ be the relatively minimal regular model of A_K over S. The technique discussed can be applied to the following cases:

(i) The special fiber A_k is an elliptic curve and is ordinary, i.e. it admits a rational point of order p (cf. Theorem 7.3),

(ii) A_k is a polygon of projective lines (cf. Theorem 7.5),

(iii) A_k is neither an ordinary elliptic curve nor a polygon of projective lines, but attains it over a finite field extension of degree prime to p (cf. Theorem 7.9).

We say that A_K has *good reduction* if A_k is an elliptic curve, *multiplicative reduction* if A_k is a polygon of projective lines and else that A_K has *additive reduction.* With this terminology, we miss the following cases: A_K has good reduction with A_k supersingular; A_K has additive reduction and attains good or multiplicative reduction only after a finite field extension of degree divisible by p; or A_K has additive reduction, but good reduction after a finite field extension with supersingular special fiber.

Note that in our considerations, we only analyzed the torsion structure of a specific abelian fibration. In the case of elliptic fibrations, the torsion structure is invariant under choosing elliptic fibrations over S with isomorphic generic fiber, cf. Proposition 6.12.

Outline of structure. In the first chapter, we set up notation and define the basic objects we are working with. We collect facts about elliptic fibrations and reprove the canonical bundle formula in a slightly more general setting than Bombieri and Mumford [7] did.

The second chapter is dedicated to the study of the jumping numbers n_i, the smallest positive integer such that $\operatorname{ord}(\mathscr{O}_{n_iF}(F))$ is equal to νp^i, and we give a first description of the torsion part in the cohomology of an elliptic fibration.

To give another description of the torsion part, we construct abelian fibrations as certain quotients in Chapter 3. Section 3.1 gives details on the notion of torsors and sets up a one-to-one correspondence between isomorphism classes of torsors under an abelian variety A_K and the first Galois cohomology group $H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), A_K(K^{\mathrm{sep}}))$ via constructing the torsor X_K as a quotient of A_K by G. To generalize this method to abelian fibrations over a complete discrete valuation ring in Section 3.3, we introduce the notion of models in Section 3.2 and cite results from the literature on the existence of certain models we want to use in Section 3.3.

Given an abelian fibration $X \to S$ over a complete discrete valuation ring, we study its behaviour under base change in Chapter 4. The first section examines the multiplicity of the closed fiber, which is used in the second section to analyze when taking a separable field extension of the function field K of S induces an étale cover. It turns out that there is a finite maximal field extension inducing an étale cover. It coincides with the one obtained in Section 3.3. Moreover, it is an invariant of the generic fiber in the case of elliptic curves.

Chapter 5 and Chapter 6 set up tools to draw information out of the quotient construction from Section 3.3: In Section 5.1, we recall the notion of group cohomology and standard facts used later. In Section 5.2, higher ramification groups are introduced, of which we compute the first cohomology group in Section 5.3 in several cases. These are often exactly the cohomology groups which reflect the torsion in $R^1 f_* \mathscr{O}_X$ for an abelian fibration $f\colon X \to S$. We sketch Sen's Theorem on the structure of $H^1(G, R')$ for a wildly ramified cyclic Galois extension R' over R of degree p^n. In Chapter 6, we set up spectral sequences for a space on which a group acts in the first section and explain in the second section some special cases how to apply these.

The application of the spectral sequences on the quotients constructed in Section 3.3 is done in Chapter 7. In the first section, we treat the cases of good reduction and multiplicative reduction, whereas in the second section we are concerned with additive reduction in the case of elliptic curves. This chapter collects the main results of this thesis.

For the major part of this thesis, we have reduced the situation to fibrations over a complete discrete valuation ring. In the first section of Chapter 8, we note that often one can go the opposite way: Given a torsor X_K over the function field $K = K(S)$ of a Dedekind scheme S and finitely many elliptic fibrations over $\hat{\mathscr{O}}_{S,s}$ with field of fractions $K_s = \mathrm{Frac}(\hat{\mathscr{O}}_{S,s})$ such that the generic fibers are isomorphic to X_{K_s}, there often is an elliptic fibration $X \to S$ such that the base change $X \times_S \hat{\mathscr{O}}_{S,s}$ is isomorphic to the prescribed fibration over $\hat{\mathscr{O}}_{S,s}$. In Section 8.2, we resume the role of elliptic fibrations in the context of Kodaira classification and study the question which elliptic fibrations with wild fiber can appear in Kodaira dimension $-\infty$ and zero with respect to the invariants given in the canonical bundle formula. In particular, we reprove the list of possible hyperelliptic surfaces addressed in [7] and [51], complementing the list by the torsion structure and excluding two potentially possible cases.

Acknowledgements. This work did not sprout from nothing. First and foremost, it was my advisor Stefan Schröer who not only gave me the opportunity for this work, but also inspired many ideas originated from discussions with him.

From all the nice people at the Mathematical Institute of the Heinrich-Heine-Universität Düsseldorf which make working there pleasant, I have to mention two of them: André Schell, whom I know since we met in a preliminary course for the freshmen students and who since then is an integral part of my mathematical life; and Benedikt Schilson, for invaluable mathematical discussions, advices with regard to formalities such a PhD thesis involves and for gently protecting me against work at the institution when time was short.

I also thank Petra Simons and Ulrike Alba for organizational help and more delicious cookies than I should have enjoyed.

For financial support from the project "SCHR 671/6-1 Enriques-Mannigfaltigkeiten", I would like to thank the Deutsche Forschungsgemeinschaft. Furthermore, I profited as a member from the activities of the DFG research training group "GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology".

Of all people apart from mathematics or the institute, I am especially indebted to my parents for taking care of my children so often when writing down this thesis. And of course to my wife, for reasons too many to mention.

Chapter 1

Elliptic fibrations

The main objects of consideration in this thesis are elliptic fibrations. We therefore want to introduce them and their main properties. Yet, most of our techniques are also applicable to higher dimensions, which will lead to the notion of abelian fibration. Thus, we work in a more general setting. Luckily, it will not afford much extra work. Most of the notation and theory is taken from [45], Chapter 8-10. Sources for elliptic fibrations in the context of the Kodaira classification of surfaces are [7] and [4], Chapter 7.

Before giving the abstract definitions, we want to give concrete examples to have in mind.

Example 1.1. Consider the projective scheme $X = \mathrm{Proj}(\mathbb{Z}[X, Y, Z]/(F))$ for the homogeneous polynomial $F = Y^2Z + YZ^2 - X^3 - XZ^2$ over $S = \mathrm{Spec}(\mathbb{Z})$. One recognizes immediately that F is a homogeneous Weierstrass equation, which over a field defines an elliptic curve (granted that the discriminant is non-zero). And indeed, taking the fiber over the generic point η of S yields

$$X_\eta = \mathrm{Proj}(\mathbb{Q}[X, Y, Z]/(Y^2Z + YZ^2 - X^3 - XZ^2)),$$

an elliptic curve over $\mathbb{Q}$. The smoothness of the fiber spreads to an open subset of S because X is proper and flat over S, cf. [19], Théorème 12.2.4. Taking the fiber at the closed point corresponding to the prime $p \in \mathbb{Z}$ gives

$$X_p = \mathrm{Proj}(\mathbb{F}_p[x, y, z]/(Y^2Z + YZ^2 - X^3 - XZ^2)).$$

To see if this fiber is smooth and thus an elliptic curve, we study the discriminant Δ: It is a polynomial in the coefficients of the Weierstraß equation and X_p is smooth if and only if $\Delta \neq 0$. One computes $\Delta = -7 \cdot 13$, so X_p is smooth over $\mathbb{F}_p$ if and only if $p \neq 7, 13$. Thus, the fibers X_7 and X_{13} are not smooth, hence X is not smooth over S. But X is regular, as one only has to check at the non-smooth points over $\mathbb{F}_7$ and $\mathbb{F}_{13}$. The point at infinity, i.e. $(0 : 1 : 0)$, is always regular, so we apply the Jacobian Criterion to the affine chart $D_+(Z) = \mathrm{Spec}(\mathbb{Z}[x, y])/(y^2 + y - x^3 - x)$ to see that there are only two non-smooth points.

They correspond to the maximal ideals $(7, x-4, y-3)$ and $(13, x+2, y-6)$. Denoting $\mathfrak{m} = (7, x-4, y-3)$ and $f = y^2 + y - x^3 - x$ as well as $A = \mathbb{Z}[x, y]$, we transform $\mathfrak{m}$ to $(7, 0, 0)$ via the isomorphism

$$A_{\mathfrak{m}} \longrightarrow A_{(7,0,0)}, \quad x \longmapsto x+4, \quad y \longmapsto y+3.$$

The image of f under the isomorphism is $y^2 + 7y - x^3 - 12x^2 - 49x - 56$, which is not contained in $(7, x, y)^2$. Hence, $A_{\mathfrak{m}}$ is regular by [45], Corollary 4.2.12. Moreover, we see that the fiber over the prime 7 is given by the equation $y^2 - x^2(x+5)$ and hence a curve with a node. We treat $(13, x+2, y-6)$ in a similar way, again obtaining a curve with a node above 13. So X is almost an elliptic curve over S, with finitely many singular fibers (also see Figure 1).

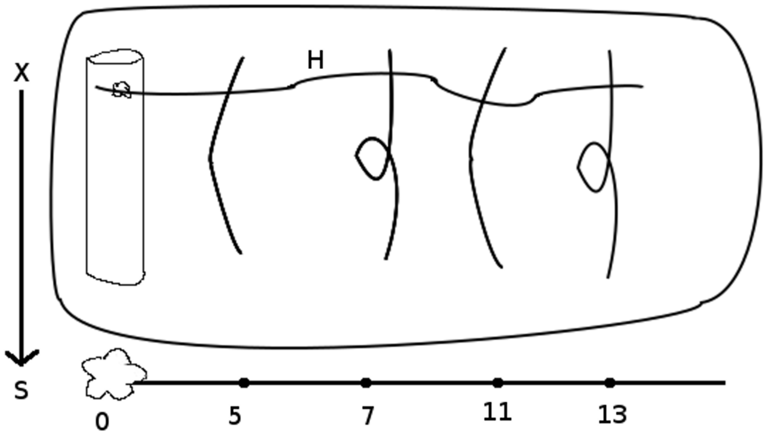


Figure 1: Schematic picture of X over S with horizontal divisor H corresponding to a rational point on the generic fiber

We will say that $X_{\mathbb{Q}}$ has *bad reduction* over these points (here $p = 7$ and $p = 13$), and otherwise *good reduction*. Note that our considerations were local over the base, so that we may replace $\mathbb{Z}$ by the p-adic integers $\mathbb{Z}_p$ for any prime p.

The next example already appeared in [31], Example 8.1, which we now elaborate. It is a motivation in Chapter 3 to construct abelian fibrations as quotients:

Example 1.2. Let k be an algebraically closed field of characteristic $p > 0$ and $S \to \mathbb{P}^1_k$ a finite cyclic Galois cover of degree p^n, totally ramified over the point $\infty \in \mathbb{P}^1_k$. That is, we want $S \to \mathbb{P}^1_k$ to be finite, surjective and étale outside of ∞, and the function field extension $K' = K(S)$ over $K = K(\mathbb{P}^1_k)$ is a finite cyclic Galois extension of degree p^n. Moreover, there lies only one point in S over ∞. Any element of the Galois group $G = \mathrm{Gal}(K'/K)$ extends to a morphism of an open subscheme of S. Moreover, as S is normal, it uniquely extends to all of S, yielding an automorphism of S.

Now take an elliptic curve E over k that is ordinary, so that it has a k-rational point P of order p^n. Denoting by σ a generator of G, we obtain an action of G on $S \times_k E$ via

$$\sigma\colon S \times_k E \longrightarrow S \times_k E, \quad (u, Q) \longmapsto (\sigma(u), Q + P).$$

The action is compatible with the action of G on S, that is, the projection $\mathrm{pr}\colon S \times_k E \to S$ is G-equivariant. Taking the quotients $X = (S \times_k E)/G$ and $S/G = \mathbb{P}^1_k$ (which exist because G is finite and the schemes are projective), we obtain the following commutative diagram

$$\begin{tikzcd} S \times_k E \arrow[r, "q"] \arrow[d, "\mathrm{pr}"'] & X \arrow[d, "f"] \\ S \arrow[r] & \mathbb{P}^1_k. \end{tikzcd}$$

Note that f is induced by the projection pr via the universal property of the quotient. Furthermore, the quotient morphism q is étale as G acts free on the closed points, cf. [56], Theorem in Chapter II, Section 7. Therefore, X is a regular and integral surface over $\mathbb{P}^1_k$. If $a \in \mathbb{P}^1_k$ is a closed point other than ∞, with $s_1, \ldots, s_{p^n} \in S$ lying above a, the fiber X_a is given by $((S \times_k E)_{s_1} \dot{\cup} \cdots \dot{\cup} (S \times_k E)_{s_{p^n}})/G$, the identification of p^n copies of E via G, hence isomorphic to E again. But the fiber X_∞ yields a new phenomenon: We consider X_∞ as a divisor rF in X for a divisor F in X that is not the positive multiple of another one. Denote by ∞_1 the single point in S lying above $\infty \in \mathbb{P}^1_k$. Taking the pullback of ∞ in $\mathbb{P}^1_k$ to $S \times_k E$ in the two different ways of above diagram and using the étaleness of q gives $p^n(\{\infty_1\} \times E) = r(\{\infty_1\} \times E)$, hence $X_\infty = p^n F$. In fact, restricting the action of G on $S \times_k E$ to $\{\infty\} \times_k E$, we see that $F = E/G$ is an elliptic curve. Therefore, X_∞ is the infinitesimal thickening of an elliptic curve. We call the number p^n its *multiplicity*. Moreover, the generic fiber X_η cannot have a rational point T, as otherwise the closure of T in X would give a *horizontal divisor*, i.e. a divisor mapping surjectively to $\mathbb{P}^1_k$ (like in Figure 1). By intersection theory on surfaces, the multiplicity of X_∞ has then to be one, a contradiction.

To set up the notion of fibration we use in this thesis, we specify the properties of the base schemes we are going to use:

Definition 1.3. We call a connected, noetherian, normal scheme S of dimension one a *Dedekind scheme.*

The naming comes from a close relation to Dedekind rings: If $\mathrm{Spec}(R)$ is an open affine subscheme of S, the ring R must be an integral, integrally closed noetherian ring of Krull dimension one, which is to say a Dedekind ring. On the other hand, if a connected, quasi-compact scheme S can be covered by the spectra of Dedekind rings of Krull dimension one, it will be a Dedekind scheme. Some authors allow Dedekind rings and Dedekind schemes to have dimension zero, but this is not the setup we are going to study.

From its definition, it immediately follows that S is integral and regular. These schemes will usually play the role of the base scheme S of our fibration. We are mostly interested in the following cases:

Example 1.4. (global case) S is a connected regular projective curve over a field k. These curves are in one-to-one correspondence to finite field extensions of $k(T)$ (cf. [15], Theorem 15.22). This case will be of interest when we look at elliptic fibrations in the Kodaira classification. Usually, S will be either the projective line or an elliptic curve over an algebraically closed field.

Example 1.5. (local case) S is the spectrum of a discrete valuation ring R with field of fractions K and algebraically closed residue field k. If we have a global Dedekind scheme, localizing gives a local Dedekind scheme. We will also often reduce to the case that R is complete, which can be achieved by completion. Then, by [66], Chapter II, §4-§6, one has the following powerful structure theorem: If $\mathrm{char}(R) = \mathrm{char}(K)$, then R is isomorphic to $k[[T]]$. If the characteristic is different, then one can uniquely embed the ring of Witt vectors $W(k)$ into R such that one obtains the identity on the residue fields. Moreover, R is a free $W(k)$-module of finite rank e and $A = W(k)[\pi]$, where π satisfies the Eisenstein equation $\pi^e + b_{e-1}\pi^{e-1} + \cdots + b_0 = 0$, with b_i being divisible by p, but b_0 is not divisible by p^2.

Having clarified over which base schemes we work, we come to the notion of fibration:

Definition 1.6. Let X be a normal, integral scheme and S be a Dedekind scheme. We call a morphism of schemes $f\colon X \to S$ a *fibration* if it is proper and if the canonical morphism $\mathscr{O}_S \to f_*\mathscr{O}_X$ is an isomorphism. We call the fibration *of dimension* n if each fiber is equidimensional of dimension n.

Note that in our situation, the isomorphism $\mathscr{O}_S \to f_*\mathscr{O}_X$ implies that f is surjective, and this is equivalent to the flatness of f by [15], Proposition 14.14. Moreover, f has geometrically connected fibers, cf. [15], Theorem 12.69. Actually, the equidimensionality of the fibers holds automatically according to [45], Lemma 8.3.3.

Usually, fibrations in the above sense are far too general to get a good understanding up to isomorphism of it, but we defined them this way to have a more flexible notion than the ones we are actually interested in, e.g. when changing the base (though one usually also has to normalize to get a fibration). We are interested in the case that the generic fiber becomes isomorphic to an abelian variety over a separable closure. This is equivalent to the fact that the generic fiber is a *torsor* under its Albanese variety, see Proposition 3.3. Note that if the dimension of X_η is one, the base change $X_\eta \otimes_K K'$ is isomorphic to an elliptic curve for some field extension K' over K. Here are the objects of our main interest:

Definition 1.7. We call a fibration $f\colon X \to S$ an *abelian fibration* if X is regular and the generic fiber X_K is a torsor under an abelian variety. If $\dim(X_\eta) = 1$, we call it an *elliptic fibration*. We also say that the fibration f is projective if the morphism f is projective.

We already saw two examples for an elliptic fibration, Example 1.1 and Example 1.2. To prove the condition on the generic fiber in the latter example, we refer to Remark 1.15. Moreover, the surface $S \times_k E$ over S in Example 1.2 is also an elliptic fibration, a *trivial* one. Note that the naming "abelian fibration" or "elliptic fibration" is misleading in the sense that the generic fiber does not have to be an abelian variety or elliptic curve, see Example 1.2. Furthermore, we want to remark that the generic fiber X_η of an abelian fibration $f\colon X \to S$ becomes smooth over K after base change, so $X_\eta \to K$ must itself be smooth by descent (cf. [20], Corollaire 17.7.3) and thus geometrically integral. Other important, though here not thoroughly treated, objects are *quasi-elliptic fibrations*: They are fibrations $f\colon X \to S$ with X regular and X_η a geometrically integral, but singular curve. They only appear in characteristic two and three. Similarly to quasi-elliptic fibrations, one may define in general a *genus-g-fibration* for an integer $g \geq 0$ as a fibration of dimension one for which the generic fiber is a geometrically connected smooth projective curve of genus g. An elliptic fibration is then nothing else than a genus-1-fibration. If one does not want to specify the genus of the generic fiber, one simply speaks of a *regular fibered surface*. Let us recall that by a theorem of Lichtenbaum [42], Chapter I, Theorem 2.8, a regular fibered surface over an *affine* Dedekind scheme is always projective. The same is true if f is a morphism of schemes proper over an algebraically closed field: By the Theorem of Zariski–Goodman (e.g. [4], Theorem 1.28), X must be projective over k, and therefore f is projective as projectivity satisfies the cancellation property.

Given an elliptic fibration or more generally a regular fibered surface X, one may always blow up a closed point a in X to get a slightly different regular fibered surface X'. The exceptional divisor $E = X_a$ is a (-1)-*curve*, that is, E is isomorphic to a projective line $\mathbb{P}^1_{\kappa(a)}$ with $\kappa(a) = H^0(E, \mathscr{O}_E)$ and has self-intersection $(E \cdot E) = -[\kappa(a)\colon k]$ (cf. [45], Proposition 9.2.5). In particular, if k is algebraically closed, the self-intersection is -1, which explains the name. Conversely, Castelnuovo's Criterion states that given a (-1)-curve on a closed fiber of X, there exists a unique morphism g from X to a regular fibered surface Y that is an isomorphism on $X \setminus E$ and maps E to a point. That is, X is obtained from Y by blowing up at $g(E)$. As in the classification of algebraic surfaces, one wants to study "minimal" objects, that is to say regular fibered surfaces that are not obtained by blowing up.

Definition 1.8. We call the regular fibered surface $X \to S$ a *relatively minimal fibered surface* if there is no (-1)-curve with support in a closed fiber.

In their foundational paper series [55], [7] and [6], Bombieri and Mumford transferred the classification of algebraic surfaces over an algebraically closed field of characteristic zero to the positive characteristic case. In shorthand, there are four different classes of algebraic surfaces according to their *Kodaira dimension*. This dimension is defined in terms of the growth of the plurigenera $P(n) = \dim_k(H^0(X, \omega_X^{\otimes n}))$ of the canonical bundle of X, where $n \in \mathbb{N}$. Iitaka showed in [29] that there are polynomials Q_1, Q_2 of the same degree $d \in \{-\infty, 0, 1, 2\}$ with positive leading coefficient such that $Q_1(n) \leq P(n) \leq Q_2(n)$ holds for large n. One then defines $\kappa(X) = d$ as the *Kodaira dimension* of X. As $P(n)$ is a birational invariant for smooth surfaces, so is $\kappa(X)$.

Surfaces of Kodaira dimension two remain rather general without additional assumptions, but the other ones have a quite explicit structure. In particular, every minimal regular surface of Kodaira dimension one without (-1)-curve is a relatively minimal elliptic surface (or quasi-elliptic surface, which appear only in characteristic two and three), and they also appear in Kodaira dimension $-\infty$ and zero (but not in Kodaira dimension two). We will give an overview in Chapter 8. In characteristic zero, Kodaira showed in [37], Theorem 12, that the canonical bundle has a very explicit form in terms of its singular fibers, namely

$$\omega_X = f^*(\mathscr{L}) \otimes \mathscr{O}_X\Big(\sum_{s \neq \eta}(m_s - 1)F_s\Big),$$

where $\mathscr{L}$ is a line bundle on S and m_s is the multiplicity of the closed fiber X_s, i.e. m_s is the greatest integer such that $X_s = m_s F_s$ holds for a Weil divisor F_s. This in turn also defines F_s. The infinite divisor sum is actually finite: As mentioned in Example 1.1, the set of points $s \in S$ such that X_s is smooth is open in S by the properness and flatness of f, see [19], Théorème 12.2.4. Hence, there are only finitely many *singular* fibers X_s with multiplicity $m_s > 1$.

In characteristic $p > 0$, Bombieri and Mumford gave a very similar formula in their article [7]. The only obstacle that comes when considering fibrations $f\colon X \to S$ in positive characteristic is the appearance of torsion in the higher direct image $R^1 f_* \mathscr{O}_X$. More generally, looking at general fibrations $f\colon X \to S$, one has a for any $i \geq 0$ a decomposition of $R^i f_* \mathscr{O}_X$ into the sum of a locally free sheaf $\mathscr{L}_i$ of rank $n_i = \dim_K H^i(X_\eta, \mathscr{O}_{X_\eta})$ and some torsion sheaf $\mathscr{T}_i$ supported on finitely many points: The sheaf $R^i f_* \mathscr{O}_X$ is coherent due to the properness of f, cf. [16], Théorème 3.2.1. As cohomology commutes with flat base change, there is a canonical isomorphism $(R^i f_* \mathscr{O}_X)_\eta \to H^i(X_\eta, \mathscr{O}_{X,\eta}) = \mathscr{O}_{X,\eta}^{\oplus n_i}$ of $\mathscr{O}_{X,\eta}$-modules. This isomorphism extends to an open subset of S (cf. [15], Proposition 7.27), which is also dense due to the irreducibility of S. Thus, $R^i f_* \mathscr{O}_X$ is free of rank n_i except at finitely many closed points. After localizing to such a point, $R^i f_* \mathscr{O}_X$ becomes a finitely generated module over a local Dedekind ring, i.e. a discrete valuation ring. By the structure theorem of finitely generated modules over discrete valuation rings, it must be isomorphic to the sum of a free part and a torsion part.

Coming back to elliptic fibrations, there is only one torsion group of interest, as $\mathscr{T}_0 = 0$

by the very definition of fibration and $R^i f_* \mathscr{O}_X = 0$ for $i \geq 2$ by dimension reasons. The main purpose of this thesis is to study the torsion structure of $\mathscr{T} = \mathscr{T}_1$. A fiber over a point $s \in S$ such that $\mathscr{T}_s$ is non-trivial will be called *wild fiber*. Else, it will be called a *tame fiber*. We are now able to state the canonical bundle formula for arbitrary characteristic. Bombieri and Mumford stated it in [7] in the case that S is a proper k-scheme. Yet, their argument works in our setting of S being an arbitrary Dedekind scheme, including the mixed characteristic case:

Theorem 1.9. *Let $f\colon X \to S$ be a relatively minimal projective elliptic or quasi-elliptic fibration and let $R^1 f_* \mathscr{O}_X = \mathscr{L} \oplus \mathscr{T}$. Denote the length of $\mathscr{T}$ by l. Then*

$$\omega_{X/S} = f^*(\mathscr{L}^\vee) \otimes \mathscr{O}_X\Big(\sum_{s \neq \eta} a_s F_s\Big)$$

holds, where

(i) $X_s = m_s F_s$ *holds as Weil divisors, and m_s is the greatest common divisor of multiplicities occurring in the decomposition of X_s into prime divisors,*

(ii) a_s *is an integer between 0 and $m_s - 1$, with $a_s = m_s - 1$ if X_s is tame and $\kappa(s)$ algebraically closed,*

(iii) $\deg(\mathscr{L}) = -l - \chi(\mathscr{O}_X)$ *when S is a projective curve over a field k.*

Moreover, if X and S are projective over a field k, then the dualizing sheaf ω_X with respect to $X \to \mathrm{Spec}(k)$ is of the form $\omega_X = \omega_{X/S} \otimes f^*(\omega_S)$, where ω_S is the dualizing sheaf with respect to $S \to \mathrm{Spec}(k)$.

Proof. We start by noting that a canonical divisor $K_{X/S}$ has no horizontal part, i.e. every prime divisor sitting in $K_{X/S}$ is contained in some fiber over a closed point. This follows from the equality

$$\omega_{X/S}|_{X_\eta} = \omega_{X_\eta} = \mathscr{O}_{X_\eta},$$

where the first equality stems from the fact that canonical sheaves commute under flat base change (cf. [45], Theorem 6.4.9) and the second equation from the fact that the canonical sheaf of a genus-one curve is trivial, cf. [45], Example 7.3.35.

Now we apply intersection theory on fibered surfaces and observe that

$$\begin{aligned}(K_{X/S} \cdot X_s) &= 2 \dim_K H^1(X_\eta, \mathscr{O}_{X_\eta}) - 2 = 0,\\ (K_{X/S} \cdot D) &\geq 0\end{aligned}$$

holds for any closed fiber X_s and prime divisor $D \subset X_s$ (e.g. [45], Proposition 9.1.35 and Proposition 9.3.10(b)). Note that the inequality uses the relative minimality of f. Let us write $X_s = \sum d_i D_i$ and assume that the support of $K_{X/S}$ lies in X_s. This is justified because the intersection number of prime divisors in different fibers is zero. From this, we deduce

$$0 = (K_{X/S} \cdot X_s) = \sum \underbrace{d_i}_{>0} \underbrace{(K_{X/S} \cdot D_i)}_{\geq 0},$$

hence $(K_{X/S} \cdot D) = 0$ for any prime divisor D in X_s. Thus, writing $K_{X/S} = \sum a_i D_i$, we obtain

$$(K_{X/S} \cdot K_{X/S}) = \sum_i a_i (K_{X/S} \cdot D_i) = 0.$$

Now taking intersection numbers on the prime divisors of X_s gives a negative semi-definite bilinear form (cf. [45], Theorem 9.1.23) and X_s is connected for every closed point in S, so we deduce from $(K_{X/S} \cdot K_{X/S}) = 0$ that the divisor $K_{X/S}$ is a multiple of X_s.

Therefore, we may write $\omega_{X/S} = f^*\mathscr{M} \otimes \mathscr{O}_X\big(\sum_{s \neq \eta} a_s F_s\big)$ as in the statement. We now want to show that $\mathscr{M} = \mathscr{L}^\vee$: As a first step, we argue that $f_*\omega_{X/S} = \mathscr{M}$. This follows from the projection formula if the canonical map $\mathscr{O}_S \to f_*\big(\mathscr{O}_X\big(\sum_{s\neq\eta} a_s F_s\big)\big)$ is bijective. This can be checked locally, so we drop the index s in our notation. From the inclusions

$$f_*(\mathscr{O}_X) \subset f_*\big(\mathscr{O}_X(aF)\big) \subset f_*\big(\mathscr{O}_X((m-1)F)\big),$$

it suffices to check the case $a = m-1$. As $mF = X_s$ for a closed point $s \in S$ holds and $S = \operatorname{Spec}(R)$ is now a local Dedekind scheme, R is a principal ideal domain and we obtain $\mathscr{O}_X(mF) = f^*\mathscr{O}_S(s) \simeq f^*\mathscr{O}_S = \mathscr{O}_X$. Again using the projection formula, it suffices to show $f_*\mathscr{O}_X(-F) = \mathscr{O}_S$. Certainly, this is an inclusion, so $f_*\mathscr{O}_X(-F)$ can be considered as a finitely generated submodule of R, i.e. an ideal. As R is a principal ideal domain, $f_*\mathscr{O}_X(-F)$ has a single generator and is isomorphic to R as an R-module, which proves the claim.

It remains to show that $\mathscr{M} = \mathscr{L}^\vee$. We therefore note that $\omega_{X/S}$ coincides with the first relative dualizing sheaf (cf. [45], Theorem 9.4.32) and hence we may apply Grothendieck duality:

$$f_*\omega_{X/S} = f_*\mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \omega_{X/S}) = f_*\mathcal{H}om_{\mathscr{O}_S}(R^1 f_*\mathscr{O}_S, \mathscr{O}_S) = \mathcal{H}om_{\mathscr{O}_S}(\mathscr{L} \oplus \mathscr{T}, \mathscr{O}_S).$$

As the dual of the torsion part is trivial, we deduce $f_*\omega_{X/S} = \mathscr{L}^\vee$. Then the claim $\omega_X = \omega_{X/S} \otimes f^*(\omega_S)$ is exactly the adjunction formula for canonical sheaves, cf. [45], Theorem 6.4.9(a).

We now turn our attention to the degree of $\mathscr{L}$, assuming that X and S are projective over a field. Applying the Leray–Serre spectral sequence yields the exact sequence

$$0 \longrightarrow H^1(S, \mathscr{O}_S) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^0(S, R^1 f_*\mathscr{O}_X) \longrightarrow 0$$

as well as $H^2(X, \mathscr{O}_X) = H^1(S, R^1 f_* \mathscr{O}_X)$. Putting these identities and and the equality $\dim_k H^0(S, \mathscr{O}_S) = \dim_k H^0(X, \mathscr{O}_X)$ into the definition of $\chi(\mathscr{O}_X)$ gives

$$\chi(\mathscr{O}_X) = \chi(\mathscr{O}_S) - \chi(\mathscr{L}) - l = -\deg(\mathscr{L}) - l.$$

To prove that X_s being tame implies that $a_s = m_s - 1$, we first observe that X_s being tame is equivalent to $\dim_{\kappa(s)} R^1 f_* \mathscr{O}_X \otimes \kappa(s) = \dim_{\kappa(s)} H^1(X_s, \mathscr{O}_{X_s}) = 1$ according to the theorem on semi-continuity and base change (cf. [17], Section (7.7), also restated in Theorem 2.9). We will prove in Proposition 2.4 that m_s decomposes into $m_s = \nu_s \cdot p^{e_s}$, where ν_s is the order of $\mathscr{O}_X(F_s)$ in $\operatorname{Pic}(F_s)$. The dimensions $\dim_{\kappa(s)} H^1(nF_s, \mathscr{O}_{nF_s})$ are monotone increasing with $n \geq 1$ and the first number $n \geq 1$ such that $\dim_{\kappa(s)} H^1(nF_s, \mathscr{O}_{nF_s}) > 1$ is $n = 1 + \nu_s$ by Lemma 2.5. Therefore, $m_s < 1 + \nu_s$, which is to say $m_s = \nu_s$. One now has the isomorphism

$$\mathscr{O}_X\big((a_s + 1)F_s\big)|_{F_s} = \omega_{F_s/\kappa(s)} \simeq \mathscr{O}_{F_s},$$

where the first equality is the adjunction formula (e.g. see [45], Theorem 9.1.37) and the second one Lemma 2.5 or [55], Corollary 1, p.333 (this is where we assume $\kappa(s)$ algebraically closed). This means that $m_s = \nu_s$ divides $a_s + 1$, so $a_s = m_s - 1$. □

Example 1.10. We resume Example 1.2: Let S be the normal, proper curve given by the Artin–Schreier extension $K' = K[x]/(x^p - x - t^m)$, where $K = k(t)$ is the function field of $\mathbb{P}^1_k$ and m is a positive integer coprime to $p > 0$, the characteristic of k. Then the Galois group G of K' over K is cyclic of order p, having a generator σ that maps a root α of $f = x^p - x - t^m$ to $\alpha + 1$. The genus of S is given by

$$g(S) = \frac{1}{2}(p-1)(m-1).$$

This can be seen as follows: The curve $C = V_+(F)$ in $\mathbb{P}^2_k$ defined by the homogeneous equation $F = X^p Z^{n-p} - XZ^{n-1} - T^m Z^{n-m}$, where $n = \max(p, m)$, has function field K' and the inclusion $k[T, Z] \to k[T, Z, X]/(F)$ gives a rational map from C to $\mathbb{P}^1_k$. One easily sees that this is defined and étale away from $\infty \in \mathbb{P}^1_k$. Concatenating with the normalization $S \to C$ of C, the map extends uniquely to a morphism $S \to \mathbb{P}^1_k$, cf. [45], Corollary 4.1.17. This morphism is étale away from ∞, as $S \to C$ is an isomorphism away from ∞. Over ∞, the morphism $S \to \mathbb{P}^1_k$ is totally ramified: We denote by ν_1 the valuation on K induced by the discrete valuation ring $\mathscr{O}_{\mathbb{P}^1, \infty}$. Then the element t from the equation $x^p - x - t^m$ has value -1. Take an extension of ν_1 on K' and let ν_2 be the normalization of it. Write $\nu_2(t) = -e$. From f, we deduce that $\nu_2(\alpha)$ is negative and hence $\nu_2(\alpha) = \nu_2(\alpha+1)$ holds. Thus, we have

$$p\nu_2(\alpha) = \nu_2\left(\prod_{i=0}^{p-1} \alpha + i\right) = \nu_2(t^m) = -me.$$

As p is coprime to m, it must divide e. Hence, as K' over K is an extension of degree p, we have $e = p$ and the extension is totally ramified. We will discuss ramification theory in Section 5.2 in greater detail.

Applying the Hurwitz Formula to $S \to \mathbb{P}^1_k$ (e.g. [45], Theorem 7.4.16), we are led to the equation

$$g(S) = \frac{1}{2}(e'_\infty - 1 + 2 - 2p),$$

where $e'_\infty - 1$ can be identified with the valuation of the different, cf. [45], Remark 7.4.17 and [66], Chapter III, §7, Proposition 14. For our Artin–Schreier extension, this is equal to $(m+1)(p-1)$. This follows from the higher ramification groups of this extension (cf. Example 5.24) and a formula for the valuation of the different in terms of its higher ramification groups (cf. [66], Chapter IV, §1, Proposition 4). Hence, we obtain

$$g(S) = \frac{1}{2}(p-1)(m-1).$$

We now observe that we can calculate $\omega_{S\times_k E}$ in two different ways: On the one hand, we have $\omega_{S\times_k E} = q^*\omega_X$ as q is étale. On the other hand, we have $\omega_{S\times_k E} = \mathrm{pr}^*\,\omega_S \otimes \mathrm{pr}_2^*\,\omega_E$, together with ω_E trivial for an elliptic curve. By [4], Proposition 9.7, the Euler characteristic $\chi(\mathscr{O}_X)$ equals $\deg(q)\chi(\mathscr{O}_{S\times_k E})$. Applying the Künneth Formula gives $\chi(\mathscr{O}_{S\times_k E}) = 0$, and the canonical bundle formula from Theorem 1.9 becomes $\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(-2+l)) \otimes \mathscr{O}_X(aF)$. Hence, if we take intersection with a horizontal divisor H, we get

$$(l-2)p + a = (q^*\omega_X \cdot H) = (\mathrm{pr}^*\,\omega_S \cdot H) = 2g(S) - 2.$$

Writing $m = dp + b$ for $1 \leq b \leq p-1$ and $d \geq 0$, we deduce

$$\begin{aligned} l &= \frac{2g(S) - 2 - a}{p} + 2 = \left\lfloor \frac{2g(S)-2}{p} \right\rfloor + 2 = m - d \\ a &= 2g(S) - 2 - p\left\lfloor \frac{2g(S)-2}{p} \right\rfloor = p - b - 1. \end{aligned}$$

In particular, we have $\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(m-d-2)) \otimes \mathscr{O}_X((p-b-1)F)$.

Remark 1.11. All possible cases for the a_s in Theorem 1.9 appear, cf. Example 1.10 and [32], Theorem 3.1 or [46], Remark 8.12.

Let us assume that the generic fiber contains a rational point $P \in X_K$, i.e. the generic fiber can be considered as an "honest" elliptic curve by choosing a "zero" point. Its closure in X gives a divisor which is a section of the structure morphism. Moreover, it has intersection number $(X_s \cdot \overline{\{P\}}) = 1$ for every closed point $s \in S$ (cf. [45], Proposition 9.1.30), meaning that in every closed fiber X_s, there is at least one divisor of multiplicity $m_s = 1$. Those elliptic surfaces are also called elliptic surfaces with section, and the sections correspond to the K-rational points of X_K. We already saw in the proof of Theorem 1.9 that those elliptic surfaces do not admit torsion: If the multiplicity m_s is one, the first number n where $\dim_{\kappa(s)} H^1(nF_s, \mathscr{O}_{nF_s}) > 1$ is $n = 1 + \nu_s = 2$, and hence the dimension of the vector space $R^1 f_* \mathscr{O}_X \otimes \kappa(s) = H^1(X_s, \mathscr{O}_{X_s})$ is one. Thus, s is not in the support of $\mathscr{T}$. For elliptic surfaces with section, Kodaira gained in [36], Theorem 6.2, a good insight into the degeneration types that can appear: Recall that the fibers over an open set in S are smooth and connected, hence there are only finitely many "degenerated" fibers. Kodaira found out that there are only finitely many types of curves that can appear in characteristic zero. Néron independently described in [57] these curves algebraically if they are defined over an algebraically closed field of any characteristic. Tate gave a very elegant algorithm in [73] that determines among other invariants the degeneration type, only assuming that the residue field of the point over which the curve lies is perfect. It is an artful manipulation of the Weierstrass equation, which can also be looked up in Silverman's book [69]. There is also a generalization for non-perfect fields by Szydlo, cf. [72]. We state the classification as the following theorem, in which N_s^0 denotes the identity component of the closed fiber over s of the Néron model N of X_η. It coincides with the smooth locus of f (cf. [45], Theorem 10.25.14) and will be discussed in Section 3.2.

Theorem 1.12. *Let $f\colon X \to S$ be a relatively minimal elliptic surface with section and $s \in S$ be a closed point with algebraically closed residue field. Then the special fiber is one of the types listed in Table 1. All Kodaira symbols appear.*

In Example 1.1, every fiber over a prime unequal to 7 or 13 was an elliptic curve, hence of Kodaira type I_0. The fibers over the primes 7 and 13 were curves with a node, that is of Kodaira type I_1. In Example 1.2, all fibers are of Kodaira type I_0.

The proof of Theorem 1.9 exploits the boundaries given by intersection theory. For example, as soon as one assumes that X_s has two or more irreducible components (and for simplicity $\kappa(s)$ algebraically closed), one starts with the formula

$$2d_i = \sum_{j \neq i} d_j (D_j \cdot D_i),$$

where $X_s = \sum_i d_i D_i$. As X_η contains a rational point P, there exists a section of f, namely the closure of the rational point in X. That implies that the intersection number of X_s and $\overline{\{P\}}$ is equal to one, and therefore one may assume that $d_1 = 1$. Hence, the formula above yields $2 = \sum_{j \geq 2} d_j (D_j \cdot D_1)$. As the intersection numbers involved are non-negative, D_1 meets at most 2 other components. Assume that $r = 2$. Then $2 = d_2(D_2 \cdot D_1)$ and

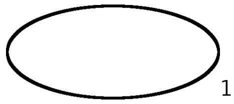
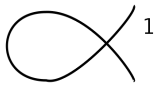
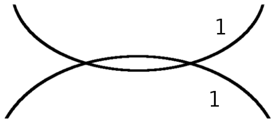
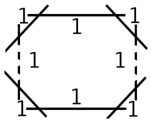
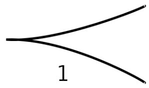
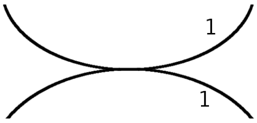
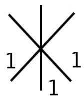
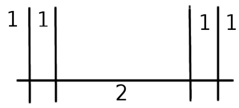
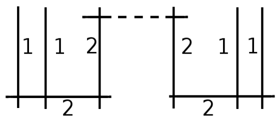
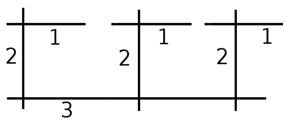
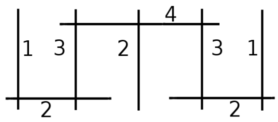
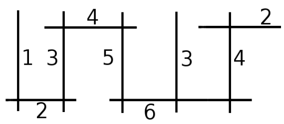
Kodaira symbol	Number of Components	N_s^0	Configuration
I_0	1	elliptic curve	
I_1	1	$\mathbb{G}_m$	
I_2	2	$\mathbb{G}_m$	
I_n	n	$\mathbb{G}_m$	
II	1	$\mathbb{G}_a$	
III	2	$\mathbb{G}_a$	
VI	3	$\mathbb{G}_a$	
I_0^*	5	$\mathbb{G}_a$	
I_n^*	5+n	$\mathbb{G}_a$	
VI^*	7	$\mathbb{G}_a$	
III^*	8	$\mathbb{G}_a$	
II^*	9	$\mathbb{G}_a$	

Table 1: Kodaira–Néron classification of singular elliptic fibers

$2d_2 = (D_1 \cdot D_2)$, that is $d_2 = 1$ and $(D_1 \cdot D_2) = 2$. Thus, either D_1 and D_2 intersect tangentially (Type III) or they meet transversally in two different points (Type I_2). For a thorough discussion, we refer to [45], §10.25 or [69], §IV.8.

Now a very natural question to ask is how the special fiber of an elliptic surface without section looks like. Take the jacobian E_K of the generic fiber X_η, so that X_η is an E_K-torsor. To E_K we associate its relatively minimal regular model E over S. As E_K has per definition a rational point as an elliptic curve, the multiplicity of its closed fiber is 1. Liu, Lorenzini and Raynaud proved in [46], Theorem 6.6, a strong relationship:

Theorem 1.13. *Let S be the spectrum of a discrete valuation ring with algebraically closed residue field and denote by m the multiplicity of the closed fiber X_s in X. Then if T denotes the Kodaira type of the fiber E_s, the fiber X_s is of Kodaira type mT.*

In particular, in the global situation that X and S are projective over an algebraically closed field k, one can apply this theorem for any fiber X_s by first base changing to $\mathrm{Spec}(\mathscr{O}_{S,s})$ and then applying the theorem.

As all but finitely many fibers of an elliptic fibration are elliptic curves, one may also ask for fibrations such that all but finitely many fibers are abelian varieties. Yasuda defined in [75] *n-abelian fiber spaces* as fibrations $f\colon X \to S$ such that X and S are smooth and projective over an algebraically closed field k and that almost all closed fibers X_s are n-dimensional abelian varieties. A fiber X_s is then called *wild* if $s \in \mathrm{Supp}(\mathscr{T}_n)$ holds and *tame* else. Yasuda obtains the following canonical bundle formula:

Theorem 1.14. *Let $f\colon X \to S$ be an n-abelian fiber space with $(K_X \cdot H^{n-1}) = 0$ for a hyperplane $H \subset X$. Denote $R^i f_* \mathscr{O}_X = \mathscr{E}_i \oplus \mathscr{T}_i$ for $1 \leq i \leq n$, where $\mathscr{E}_i$ is a locally free sheaf and $\mathscr{T}_i$ an torsion sheaf of length l_i. Then we have*

$$\omega_X = f^*(\mathscr{E}_n^\vee \otimes \omega_S) \otimes \mathscr{O}_X\Big(\sum_{i=1}^r a_i F_i\Big),$$

where

(i) $m_i F_i = X_{s_i}$ *are the finitely many multiple fibers of* f,

(ii) $0 \leq a_i \leq m_i - 1$,

(iii) $a_i = m_i - n_i$ *holds for every tame fiber* X_{s_i}, *where*

$$n_i = \min\{n \geq 1 \mid \dim_k H^0(X, \omega_{nF_i}) > 0\},$$

(iv) $\chi(\mathscr{O}_X) = \sum_{i=1}^n (-1)^i (\deg \mathscr{E}_i + l_i)$.

Remark 1.15. Every n-abelian fiber space is an abelian fibration in our sense. This can be seen as follows: The points $s \in S$ such that X_s is smooth and geometrically integral over $\kappa(s)$ form an open subset (cf. [19], Théorème 12.2.4). So shrinking the base, we may assume that every fiber is smooth and geometrically integral. Thus, its Picard scheme $\mathrm{Pic}_{X/S}$ exists (cf. [34], Theorem 4.18.1) and its degree zero component $\mathrm{Pic}^0_{X/S}$ is an abelian variety (cf. [34], Proposition 5.20). This in turn allows us to consider the morphism $X \to \mathrm{Alb}^1_{X/S}$, where $\mathrm{Alb}^1_{X/S}$ is a torsor under the Albanese $\mathrm{Alb}^0_{X/S}$. The morphism is universal in the sense that every morphism from X to a torsor under an abelian variety factors through it. Its construction commutes with base change (cf. [22], n°236, Théorème 3.3), so it is an isomorphism over each point $s \in S$ such that X_s is an abelian variety. In particular, by [16], Proposition 4.6.7 (ii), this spreads to an isomorphism over an open subset of S. Hence, X_η is isomorphic to $\mathrm{Alb}^1_{X_\eta/K(S)}$, the torsor under the abelian variety $\mathrm{Alb}^0_{X_\eta/K(S)}$. Note that we will discuss this sort of construction further in Section 3.1.

Chapter 2

The torsion structure: First description

In this chapter, we give one way for studying the torsion part $\mathscr{T} \subset R^1 f_* \mathscr{O}_X$ of an elliptic fibration $f\colon X \to S$. In fact, it suffices to do this for the spectrum of a complete discrete valuation ring: Let $s \in S$ be a closed point, $\widehat{S} = \operatorname{Spec}(\widehat{\mathscr{O}}_{S,s})$ and consider the base change $X_{\widehat{S}} = X \times_S \widehat{S}$, where the completion is taken with respect to the unique maximal ideal of $\mathscr{O}_{S,s}$. Then $\widehat{f}\colon X_{\widehat{S}} \to \widehat{S}$ is proper as the base change of f. Furthermore, the canonical morphism $\mathscr{O}_{\widehat{S}} \to \widehat{f}_* \mathscr{O}_{X_{\widehat{S}}}$ is bijective due to the fact that cohomology commutes with flat base change, cf. [26], Proposition 9.3. Hence, $\widehat{f}$ is a fibration.

To see that $\widehat{f}$ is an elliptic fibration, we note that the generic fiber is isomorphic to $X_{\widehat{K}}$, which is again a torsor under an abelian variety as this notion is stable under base change. As $X_{\widehat{S}}$ is an excellent scheme and its regular locus is open (cf. Section 4.1), one checks regularity at closed points. Denoting $p\colon X_{\widehat{S}} \to X$, the map on stalks $\mathscr{O}_{X,p(x)} \to \mathscr{O}_{X_{\widehat{S}},x}$ induces a bijection on their completions for closed points $x \in X_{\widehat{S}}$ by [71], Tag 0BG6. A noetherian local ring is regular if and only if its completion is regular, so regularity follows.

For future reference, the following proposition states that we do not "lose" knowledge of a fiber over a closed point $s \in S$ when we work over the completion of $\mathscr{O}_{S,s}$.

Proposition 2.1. *With the notation above, $\widehat{f}\colon X_{\widehat{S}} \to \widehat{S}$ is an elliptic fibration. It is relatively minimal if and only if $f\colon X \to S$ is relatively minimal. Furthermore, the multiplicities of the closed fibers coincide. Write $\omega_{X/S} = f^*(\mathscr{L}^\vee) \otimes \mathscr{O}_X\big(\sum_{s \neq \eta} a_s F_s\big)$. Then*

$$\omega_{X_{\widehat{S}}/\widehat{S}} = \mathscr{O}_{X_{\widehat{S}}}\big(a_s \widehat{F}_s\big)$$

holds and the torsion in $R^1 f_* \mathscr{O}_{X_{\widehat{S}}}$ is isomorphic to $\mathscr{T}_s \otimes_{\mathscr{O}_{S,s}} \widehat{\mathscr{O}}_{S,s}$.

Proof. We already showed that $\widehat{f}$ is an elliptic fibration and we proceed on showing the equivalence statement. Assume that $X \to S$ is not relatively minimal. Then there is a regular fibered surface Z such that X is obtained from Z by blowing up a regular closed point. The blow-up morphism $X \to Z$ commutes with flat base change, so that $X \times_S \widehat{S} \to Z \times_S \widehat{S}$ is again the blow-up in a regular closed point. Hence $X_{\widehat{S}}$ is not relatively minimal. Conversely, assume that we have a (-1)-curve $E \subset X_{\widehat{S}}$. Denote its image in X

under the projection $p\colon X_{\widehat{S}} \to X$ by T. As $\omega_{X_{\widehat{S}/\widehat{S}}} = p^*\omega_{X/S}$ holds by flat base change (cf. [45], Theorem 6.4.9(b)) and p restricted to E is a finite morphism over $\kappa(s)$, we have the degree formula

$$[K(E)\colon K(T)] \deg_{\kappa(s)} \omega_{X/S}|_T = \deg_{\kappa(s)} \omega_{X_{\widehat{S}}/\widehat{S}}|_E < 0.$$

Hence, T is a (-1)-curve (e.g. [45], Proposition 9.3.10(b)).

We already used that $\omega_{X_{\widehat{S}/\widehat{S}}} = p^*\omega_{X/S}$ holds. As $\operatorname{Pic}(S) = 0$, we only have to check for the formula on $\omega_{X_{\widehat{S}}/\widehat{S}}$ that the multiplicity of the fiber is stable under base change. But this follows from the isomorphism $\widehat{\mathscr{O}}_{X_{\widehat{S}},x} = \widehat{\mathscr{O}}_{X,p(x)}$ at any closed point x in $X_{\widehat{S}}$. The statement on the torsion is due to the fact that cohomology commutes with flat base change. $\square$

It thus suffices to study the case that S is the spectrum of a complete discrete valuation ring R with field of fractions K. For the rest of this chapter, we assume this situation together with the assumption that the residue field k is algebraically closed if not stated otherwise. In the notation, we will omit the base point s, i.e. we will write $m = m_s$, $a = a_s$ and so on. To get more information on the multiplicity m of the special fiber $X_k = mF$, we make the following observation: The invertible sheaf $\mathscr{O}_X(F)$ restricted to F as a subscheme of X is still invertible, called the *normal bundle of* F, and its order must divide m, as $\mathscr{O}_X(mF) = \mathscr{O}_X(X_k) = f^*\mathscr{O}_S(s) \simeq f^*\mathscr{O}_S = \mathscr{O}_X$ holds.

Lemma 2.2. *The order ν of $\mathscr{O}_X(F)|_F$ in* $\operatorname{Pic}(F)$ *divides m.*

Moreover, the same holds for all orders of $\mathscr{O}_X(F)|_{nF}$ in $\operatorname{Pic}(nF)$ for $n > 0$. The map $n \mapsto \operatorname{ord}(\mathscr{O}_X(F)|_{nF})$ is monotone increasing as the restriction $\mathscr{O}_X(F) \to \mathscr{O}_{nF}(F)$ factors over $\mathscr{O}_{lF}(F)$ for any $l \geq n$. The supremum should equal m. This is in fact the case, as follows from [62], Lemma 6.4.4:

Lemma 2.3. *Let $f\colon X \to S =$ $\operatorname{Spec}(R)$ *be a proper, flat morphism with R a discrete valuation ring. Then there exists a positive integer n such that all $\mathscr{L} \in$ $\operatorname{Pic}(X)$ *which satisfy $\mathscr{L}|_{lmF} \simeq \mathscr{O}_{lmF}$ and $\mathscr{L}|_{X_\eta} \simeq \mathscr{O}_{X_\eta}$ for $l \geq n+1$, are trivial. In particular, we have* $\operatorname{ord}(\mathscr{L}|_{lmF}) = \operatorname{ord}(\mathscr{L}|_{(n+1)mF}) = 1$ *for all $l \geq n+1$.*

Taking $\mathscr{L} = \mathscr{O}_X(F)$ in our case and assuming that $\mathscr{O}_{nF}(\tilde{m}F) = \mathscr{O}_{nF}$ for some $\tilde{m} < m$ and all n large enough, the lemma states that $\mathscr{O}_X(\tilde{m}F)$ would be trivial in contradiction to mF being the pullback of the uniformizer of R and therefore m minimal with the property that mF is the trivial divisor.

The number n in Lemma 2.3 can be made explicit: For a quasi-coherent sheaf $\mathscr{E}$ on S, denote by $\mathbb{V}(\mathscr{E}) = \operatorname{Spec}(\operatorname{Sym}_{\mathscr{O}_S}(\mathscr{E}))$ the *quasi-coherent bundle defined by $\mathscr{E}$*. By [9], Corollary 8.1/8, the functor

$$(\mathrm{Sch}/S) \longrightarrow (\mathrm{Set}), \quad T \longmapsto H^0(X_T, \mathscr{O}_{X_T})$$

is representable by $\mathbb{V}(\mathscr{M})$ for a coherent $\mathscr{O}_S$-module $\mathscr{M}$. One then can choose n in above lemma to be the length of the annihilator ideal of the torsion submodule of $\mathscr{M}$.

In order to get a better understanding of how the multiplicity is composed, we study the order of $\mathscr{O}_{nF}(F)$ and give a connection to the length of the torsion part. This was already done by Bertapelle and Tong in [5], §3, who themselves used unpublished results by Michel Raynaud.

For notational setup, let $\pi \in R$ be a uniformizer and $\mathscr{I} = \mathscr{O}_X(-F)$. Then $\mathscr{I}^m = \pi\mathscr{O}_X$. As $\mathscr{I} = \mathscr{O}_X(F)^\vee$ and $\mathscr{O}_X(F)$ have the same order, it suffices to study the growth of the order of $\mathscr{I}|_{nF} \in \mathrm{Pic}(nF)$. We consider the following two exact sequences:

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{O}_{nF} \longrightarrow \mathscr{O}_{(n-1)F} \longrightarrow 0$$
$$0 \longrightarrow 1+\mathscr{N} \longrightarrow \mathscr{O}_{nF}^\times \longrightarrow \mathscr{O}_{(n-1)F}^\times \longrightarrow 0$$

where $\mathscr{N} \cong \mathscr{I}^{n-1}/\mathscr{I}^n \subset \mathscr{O}_{nF}$. Note that $\mathscr{N}^2 = 0$, so that $x \mapsto 1+x$ gives an isomorphism $\mathscr{N} \to 1+\mathscr{N}$. This induces an isomorphism $\beta\colon H^1(nF,\mathscr{N}) \to H^1(nF, 1+\mathscr{N})$ on cohomology groups such that for $\tilde{n} = n-1$, the diagram

$$\begin{tikzcd}
H^0(\tilde{n}F, \mathscr{O}_{\tilde{n}F}) \arrow[r, "\delta"] & H^1(nF,\mathscr{N}) \arrow[r] \arrow[d, "\beta"] & H^1(nF,\mathscr{O}_{nF}) \arrow[r, "\alpha'"] & H^1(\tilde{n}F,\mathscr{O}_{\tilde{n}F}) \\
H^0(\tilde{n}F, \mathscr{O}_{\tilde{n}F}^\times) \arrow[r, "\delta^\times"] & H^1(nF,1+\mathscr{N}) \arrow[r, "\gamma"] & \mathrm{Pic}(nF) \arrow[r, "\alpha"] & \mathrm{Pic}(\tilde{n}F)
\end{tikzcd}$$

satisfies $\beta(\mathrm{im}(\delta)) = \mathrm{im}(\delta^\times)$, cf. [59], §6, Proposition. As a consequence, the isomorphism β induces an isomorphism $\mathrm{coker}(\delta) \to \ker(\alpha)$ via the universal property of the cokernel. Identifying $\ker(\alpha) = \mathrm{coker}(\delta^\times)$ yields the commutative diagram of exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{im}(\delta) \arrow[r] \arrow[d, "\beta|_{\mathrm{im}(\delta)}"] & H^1(nF,\mathscr{N}) \arrow[r] \arrow[d, "\beta"] & \mathrm{coker}(\delta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{im}(\delta^\times) \arrow[r] & H^1(nF,1+\mathscr{N}) \arrow[r] & \mathrm{coker}(\delta^\times) \arrow[r] & 0.
\end{tikzcd}$$

Remembering that $\mathscr{N} \cong \mathscr{I}^{n-1}/\mathscr{I}^n$ is annihilated by π, we see that the finitely generated R-module $H^1(nF,\mathscr{N})$ is in fact a finite k-vectorspace, and hence $\mathrm{coker}(\delta) = \ker(\alpha)$ as well. If p denotes the characteristic exponent of k, every non-trivial element in $\ker(\alpha)$ is of order p for $p > 1$ and else non-torsion. So the order of $\mathscr{I}|_{nF} \in \mathrm{Pic}(nF)$ either equals the order of $\mathscr{I}|_{(n-1)F} \in \mathrm{Pic}((n-1)F)$ or is equal to $p \cdot \mathrm{ord}(\mathscr{I}|_{(n-1)F})$. Denoting by ν the order of $\mathscr{I}|_F$ in $\mathrm{Pic}(F)$, we thus see by Lemma 2.3 that $m = \nu p^e$ for some $e \geq 0$. We summarize:

Proposition 2.4. *The multiplicity m of a closed fiber X_k can be factored into $m = \nu p^e$, where ν is the order of $\mathscr{O}_F(F)$ in* $\mathrm{Pic}(F)$. *In particular, if $p = 1$, we have $m = \nu$.*

Let us assume that $p > 1$. For $i = 0, \dots, e$, let n_i denote the smallest integer $n \geq 1$ such that $\mathscr{I}|_{nF}$ is of order νp^i. Moreover, write $h^i\mathscr{F} = \dim_k H^i(X, \mathscr{F})$ for a sheaf $\mathscr{F}$ on a k-scheme X. The following lemma is from [5], Lemma 3.12.

Lemma 2.5. *The following statements hold:*

(i) For $i = 0, \dots, e$, the sheaf $\omega_{n_iF} = \omega_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n_iF)|_{n_iF}$ is isomorphic to $\mathscr{O}_{n_iF}$,

(ii) For $i = 0, \dots, e-1$, the difference $n_{i+1} - n_i$ is positive and divisible by νp^i,

(iii) The integers $n \in (n_i, n_{i+1}]$ such that $h^1\mathscr{O}_{nF} > h^1\mathscr{O}_{(n-1)F}$ are exactly those which can be written as $n = n_i + h\nu p^i$ for some integer h. Furthermore, we then have $h^1\mathscr{O}_{nF} = h^1\mathscr{O}_{(n-1)F} + 1$.

Writing $\omega_{X/S} = \mathscr{O}_X(aF)$, the first statement in the lemma reads as

$$\mathscr{O}_X((a + n_i)F)|_{n_iF} \simeq \mathscr{O}_{n_iF},$$

which means that $\nu p^i \mid a + n_i$. We therefore find an integer α_i such that $n_i = -a + \alpha_i \nu p^i$. In particular, $1 = n_0 = -a + \alpha_0 \nu$ holds, so that calculating a or α_0 is equivalent. Furthermore, we will use the positive integers $k_i = (n_{i+1} - n_i)(\nu p^i)^{-1}$ for $i = 0, \dots, e$. They are indeed integers due to the second statement of above lemma.

Remark 2.6. The sets $\{\alpha_0, \dots, \alpha_e\}, \{k_0, \dots, k_{e-1}\}$ and $\{n_0, \dots, n_e\}$ defined in Lemma 2.5 and afterwards can be calculated from each other: Apparently, this is true for the n_i and k_i due to the formula $n_{i+1} - n_i = k_i \nu p^i$ and $n_0 = 1$. The formula $n_i = -a + \alpha_i \nu p^i$ gives a direct translation between the n_i and α_i.

We will use the description of the jumping values of the order of the normal bundle by α_i to describe the torsion in the cohomology of a wild fiber. Therefore, we compose two results from [5], Lemma 3.16 and Proposition 3.17, on the length of the torsion part:

Proposition 2.7. *The length of $\mathscr{T}$ is given by $\lfloor \chi m^{-1} \rfloor$, where χ is given by*

$$\chi = (m - 1) + \sum_{i=0}^{e-1} k_i (m - \nu p^i).$$

We give another way to compute the length of the torsion part by the α_i, which seems to be nicer:

Corollary 2.8. *We have*

$$l(\mathcal{T}) = (1 - \alpha_0) + (p - 1) \sum_{i=1}^{e} \alpha_i.$$

Proof. We apply the formula $k_i = (n_{i+1} - n_i)(\nu p^i)^{-1} = \alpha_{i+1} p - \alpha_i$ to $l(\mathcal{T}) = \lfloor \chi m^{-1} \rfloor$:

$$\begin{aligned} l(\mathcal{T}) = \lfloor \chi m^{-1} \rfloor &= \left\lfloor \frac{(m-1) + \sum_{i=0}^{e-1} k_i (m - \nu p^i)}{m} \right\rfloor \\ &= \left\lfloor 1 - \frac{1}{m} + \sum_{i=0}^{e-1} k_i (1 - p^{i-e}) \right\rfloor \\ &= \left\lfloor 1 - \frac{1}{m} + \sum_{i=0}^{e-1} (\alpha_{i+1} p - \alpha_i)(1 - p^{i-e}) \right\rfloor \\ &= \left\lfloor 1 - \frac{1}{m} + \sum_{i=0}^{e-1} \alpha_{i+1} (p - p^{i+1-e}) - \sum_{i=0}^{e-1} \alpha_i (1 - p^{i-e}) \right\rfloor, \end{aligned}$$

where we expanded the term $(\alpha_{i+1} p - \alpha_i)(1 - p^{i-e})$ in the last equality. Rearranging the sum, we obtain

$$\begin{aligned} l(\mathscr{T}) &= \left\lfloor 1 - \frac{1}{m} + \alpha_e (p-1) + \sum_{i=1}^{e-1} \alpha_i (p - p^{i-e}) - \sum_{i=1}^{e-1} \alpha_i (1 - p^{i-e}) - \alpha_0 (1 - p^{-e}) \right\rfloor \\ &= \left\lfloor 1 - \frac{1}{m} - \alpha_0 (1 - \frac{\nu}{m}) + \alpha_e (p-1) + \sum_{i=1}^{e-1} \alpha_i (p - p^{i-e} - 1 + p^{i-e}) \right\rfloor. \end{aligned}$$

Now we excerpt the terms with integral value from the floor function and use the identity $\alpha_0 \nu = 1 + a$ to see

$$\begin{aligned} l(\mathscr{T}) &= \left\lfloor 1 - \alpha_0 + \frac{\alpha_0 \nu - 1}{m} + \sum_{i=1}^{e} \alpha_i (p-1) \right\rfloor \\ &= 1 - \alpha_0 + (p-1) \sum_{i=1}^{e} \alpha_i + \left\lfloor \frac{a}{m} \right\rfloor \\ &= 1 - \alpha_0 + (p-1) \sum_{i=1}^{e} \alpha_i, \end{aligned}$$

where we used $\lfloor a m^{-1} \rfloor = 0$ due to $0 \leq a < m$ in the last equation. This shows the formula. $\square$

We are now going to study the structure of the torsion in the cohomology. Recall that we work under the general assumption that S is the spectrum of a complete discrete valuation ring R and that we write $X_k = mF$ for the unique closed fiber. As $R^2 f_* \mathscr{O}_X = 0$ holds, we have a canonical isomorphism $R^1 f_* \mathscr{O}_X \otimes_R R/(\pi) \to H^1(mF, \mathscr{O}_{mF})$ by the following theorem on semi-continuity and base change (cf. [17], Section (7.7)):

Theorem 2.9. *Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes and $\mathscr{F}$ be a coherent sheaf on X, flat over S. Let $s \in S$ and $i \geq 0$, then:*

(i) If the natural map

$$\varphi^i(s)\colon R^i f_*(\mathscr{F}) \otimes \kappa(s) \longrightarrow H^i(X_s, \mathscr{F}_s)$$

is surjective, it is bijective and the same holds for all s' in some neighbourhood of s.

(ii) If $\varphi^i(s)$ is surjective, then the following are equivalent:

a) $\varphi^{i-1}(s)$ is also surjective.

b) $R^i f_*(\mathscr{F})$ is locally free in an open neighbourhood of s.

In fact, this means more generally that for $n \geq 1$, the canonical maps

$$R^1 f_* \mathscr{O}_X \otimes_R R/(\pi^n) \longrightarrow H^1(nmF, \mathscr{O}_{nmF})$$

are bijective, cf. [26], Proposition 12.5 and Proposition 12.10. So $R^1 f_* \mathscr{O}_X$ decomposes into a free part R and a torsion part $\mathscr{T}$, i.e. $R^1 f_* \mathscr{O}_X \simeq R \oplus \mathscr{T}$. We treat $R^1 f_* \mathscr{O}_X$ as a plain R-module and inductively write

$$\begin{aligned} H^1(mF, \mathscr{O}_{mF}) &= R^1 f_* \mathscr{O}_X \otimes_R R/(\pi) \simeq R/(\pi)^{\oplus x_1} \\ H^1(2mF, \mathscr{O}_{2mF}) &= R^1 f_* \mathscr{O}_X \otimes_R R/(\pi^2) \simeq R/(\pi)^{\oplus x_1 - x_2} \oplus R/(\pi^2)^{\oplus x_2} \\ &\ \ \vdots \\ H^1(nmF, \mathscr{O}_{nmF}) &= R^1 f_* \mathscr{O}_X \otimes_R R/(\pi^n) \simeq \bigoplus_{i=1}^{n-1} (R/(\pi^i))^{\oplus x_i - x_{i+1}} \oplus (R/(\pi^n))^{\oplus x_n}, \end{aligned}$$

where the integers x_i for $i \geq 0$ are inductively defined by these bijections. This comes from the fact that in each step from n to $n+1$, only the torsion summands of highest length n can increase by length, hence all new torsion summands of length $n+1$ in $H^1((n+1)mF, \mathscr{O}_{(n+1)mF})$ come at the cost of reducing the torsion summands of length n, i.e. x_n becomes the number $x_n - x_{n+1}$. So our aim is to compute these x_n. One easily sees that $x_1 = h^1 \mathscr{O}_{mF}$ and that for $n \geq 2$, the x_n are given by

$$x_n = h^1 \mathscr{O}_{nmF} - h^1 \mathscr{O}_{(n-1)mF} :$$

Just compute $h^1\mathscr{O}_{nmF} = x_1 + \ldots + x_n$ and the statement follows. An immediate observation is that x_1 gives the number of direct summands appearing in $R^1 f_*\mathscr{O}_X$ and that for nm bigger than n_e (resp. $n > \alpha_e$ due to $n_e = -a + \alpha_e m$), the equality $x_n = 1$ holds. Therefore, we have to count the number of *jumping numbers* between $(n-1)m$ and nm, that is to say, integers $l \in ((n-1)m, nm]$ such that $h^1\mathscr{O}_{(l+1)F} > h^1\mathscr{O}_{lF}$ holds. By Lemma 2.5, they are exactly of the form $l = n_i + h\nu p^i$ for some i. If $n_i \leq (n-1)m$ and $nm \leq n_{i+1}$ hold, that is to say that the *jumping distance* νp^i is constant in this range, this counting is easy: We have

$$nm - (n-1)m = m = \nu p^e = (\nu p^i)p^{e-i}, \tag{2.1}$$

i.e. if $(n-1)m$ is a jumping number, we have p^{e-i} jumping numbers in $((n-1)m, nm]$. If the first jumping number between $(n-1)m$ and nm is greater than $(n-1)m$, we also have to count the first jumping number, but let drop the jumping number that exceeds nm, so again p^{e-i} jumping numbers. However, the general situation is more complicated, having in general several increases of jumping distances to consider (i.e. several n_i lying between $(n-1)m$ and m). To handle these, we define

$$i_n = \max\{\, i \mid 0 \leq i \leq e \text{ and } n_i < nm \,\} \quad \text{and } i_0 = 0,$$

that is, the greatest integer i such that n_i is smaller than nm. Then we can write

$$h^1\mathscr{O}_{nmF} = \sum_{j=0}^{i_n - 1} k_j + h_{i_n},$$

where k_j are defined in Lemma 2.5 and h_{i_n} is the unique number satisfying

$$nm = n_{i_n} + h_{i_n}\nu p^{i_n} + r_{i_n} \text{ with } 0 \leq r_{i_n} < \nu p^{i_n}.$$

That is, we count the number of jumping numbers we have before reaching nm. From this point on, it is easy to calculate

$$x_n = \sum_{j=i_{n-1}}^{i_n - 1} k_j + h_{i_n} - h_{i_{n-1}}$$

$$\text{and} \qquad x_n - x_{n+1} = 2h_{i_n} - h_{i_{n-1}} - h_{i_{n+1}} + \sum_{j=i_{n-1}}^{i_n - 1} k_j - \sum_{j=i_n}^{i_{n+1}-1} k_j.$$

Note that if we have the equalities $i_{n-1} = i_n = i_{n+1}$, the jumping distance between $(n-1)m$ and $(n+1)m$ does not change, so that $x_n - x_{n+1} = 0$ holds. Thus, we "only" need to compute these differences until $n+1$, where $i_n = e$. We now want to compute a formula that is more compact:

We start with the equation $x_n = \sum_{j=i_{n-1}}^{i_n-1} k_j + h_{i_n} - h_{i_{n-1}}$ and substitute $k_j = \alpha_{j+1}p - \alpha_j$ as in Corollary 2.8. This gives us

$$\begin{aligned} x_n &= \sum_{j=i_{n-1}}^{i_n-1} (\alpha_{j+1}p - \alpha_j) + h_{i_n} - h_{i_{n-1}} \\ &= (p-1) \sum_{j=i_{n-1}+1}^{i_n} \alpha_j + (h_{i_n} + \alpha_{i_n}) - (h_{i_{n-1}} + \alpha_{i_{n-1}}). \end{aligned}$$

Now recalling the definition of h_{i_n}, we observe

$$\begin{aligned} h_{i_n} &= \left\lfloor \frac{nm - n_{i_n}}{\nu p^{i_n}} \right\rfloor \\ &= \left\lfloor \underbrace{\frac{nm}{\nu p^{i_n}}}_{=np^{e-i_n}} + \frac{a}{\nu p^{i_n}} - \underbrace{\frac{n_{i_n} + a}{\nu p^{i_n}}}_{=\alpha_{i_n}} \right\rfloor \\ &= \left\lfloor \frac{a}{\nu p^{i_n}} \right\rfloor + np^{e-i_n} - \alpha_{i_n} \\ &= \left\lfloor \frac{\alpha_0 - 1}{p^{i_n}} \right\rfloor + np^{e-i_n} - \alpha_{i_n}, \end{aligned}$$

where in the last line, we used the equality $1 + a = \nu\alpha_0$ to compute $a\nu^{-1} = \alpha_0 - \nu^{-1}$ and thereby

$$\left\lfloor \frac{a}{\nu p^{i_n}} \right\rfloor = \left\lfloor \frac{\lfloor a\nu^{-1} \rfloor}{p^{i_n}} \right\rfloor = \left\lfloor \frac{\alpha_0 + \lfloor -\nu^{-1} \rfloor}{p^{i_n}} \right\rfloor = \left\lfloor \frac{\alpha_0 - 1}{p^{i_n}} \right\rfloor.$$

Using the geometric series, we see $(p-1)\sum_{j=i_{n-1}+1}^{i_n} p^{e-j} = p^{e-i_{n-1}} - p^{e-i_n}$ and hence obtain

$$\begin{aligned} x_n &= (p-1) \sum_{j=i_{n-1}+1}^{i_n} \alpha_j + \left\lfloor \frac{\alpha_0 - 1}{p^{i_n}} \right\rfloor + np^{e-i_n} - \left\lfloor \frac{\alpha_0 - 1}{p^{i_{n-1}}} \right\rfloor - (n-1)p^{e-i_{n-1}} \\ &= (p-1) \sum_{j=i_{n-1}+1}^{i_n} \alpha_j + \left\lfloor \frac{\alpha_0 - 1}{p^{i_n}} \right\rfloor - \left\lfloor \frac{\alpha_0 - 1}{p^{i_{n-1}}} \right\rfloor + n(p^{e-i_n} - p^{e-i_{n-1}}) + p^{e-i_{n-1}} \\ &= (p-1) \sum_{j=i_{n-1}+1}^{i_n} (\alpha_j - np^{e-j}) + \left\lfloor \frac{\alpha_0 - 1}{p^{i_n}} \right\rfloor - \left\lfloor \frac{\alpha_0 - 1}{p^{i_{n-1}}} \right\rfloor + p^{e-i_{n-1}}. \end{aligned}$$

For reasons we will see later, we want to put the "floor function terms" into the sum. This makes us need some kind of geometric series for the floor function and p-adic expansion of integers:

Lemma 2.10. *Let x be a non-negative integer, $p > 0$ prime and $x = \sum_{i\geq 0} x_i p^i$ its p-adic expansion with $s_p(x) = \sum_{i\geq 0} x_i$. Then the following holds:*

(i) Legendre's Formula

$$\sum_{i\geq 1} \left\lfloor \frac{x}{p^i} \right\rfloor = \frac{x - s_p(x)}{p-1}.$$

(ii) For $l \geq j \geq 0$, the equality

$$\sum_{i=j+1}^{l} \left\lfloor \frac{x}{p^i} \right\rfloor = \frac{1}{p-1}\Big(\left\lfloor \frac{x}{p^j} \right\rfloor - \left\lfloor \frac{x}{p^l} \right\rfloor + s_p\Big(\left\lfloor \frac{x}{p^l} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{x}{p^j} \right\rfloor\Big)\Big)$$

holds.

(iii) For $l \geq j \geq 0$ and $d \in \mathbb{N} \setminus \{0\}$, we have

$$\sum_{i=j+1}^{l} \left\lfloor \frac{x}{dp^i} \right\rfloor = \frac{1}{p-1}\Big(\left\lfloor \frac{x}{dp^j} \right\rfloor - \left\lfloor \frac{x}{dp^l} \right\rfloor + s_p\Big(\left\lfloor \frac{x}{dp^l} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{x}{dp^j} \right\rfloor\Big)\Big).$$

(iv) For $l \geq j \geq 0$, we have

$$\left\lfloor \frac{x}{p^j} \right\rfloor = \sum_{i\geq 0} x_{j+i} p^i \quad \text{and in particular} \quad s_p\Big(\left\lfloor \frac{x}{p^l} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{x}{p^j} \right\rfloor\Big) = -\sum_{i=j}^{l-1} x_j.$$

Proof. The first assertion can be found in [3], Theorem 6.5.1. For the second one, we apply Legendre's Formula as follows:

$$\sum_{i\geq j+1} \left\lfloor \frac{x}{p^i} \right\rfloor = \sum_{i\geq 1} \left\lfloor \frac{\lfloor xp^{-j} \rfloor}{p^i} \right\rfloor = \frac{\left\lfloor \frac{x}{p^j} \right\rfloor - s_p\Big(\left\lfloor \frac{x}{p^j} \right\rfloor\Big)}{p-1}.$$

Thus, one gets

$$\begin{aligned}\sum_{i=j+1}^{l} \left\lfloor \frac{x}{p^i} \right\rfloor &= \sum_{i\geq j+1} \left\lfloor \frac{x}{p^i} \right\rfloor - \sum_{i\geq l+1} \left\lfloor \frac{x}{p^i} \right\rfloor \\ &= \frac{1}{p-1}\Big(\left\lfloor \frac{x}{p^j} \right\rfloor - \left\lfloor \frac{x}{p^l} \right\rfloor + s_p\Big(\left\lfloor \frac{x}{p^l} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{x}{p^j} \right\rfloor\Big)\Big).\end{aligned}$$

The third assertion follows directly from the second by using nested division

$$\left\lfloor \frac{x}{dp^i} \right\rfloor = \left\lfloor \frac{\lfloor xd^{-1} \rfloor}{p^i} \right\rfloor,$$

and the fourth assertion follows directly from

$$\left\lfloor \frac{x}{p^j} \right\rfloor = \left\lfloor \frac{\sum_{i\geq 0} x_i p^i}{p^j} \right\rfloor = \Big\lfloor \underbrace{\sum_{i\geq 0} x_{j+i}p^i}_{\in \mathbb{N}} + p^{-j} \underbrace{\sum_{i=0}^{j-1} x_i p^i}_{<p^j} \Big\rfloor = \sum_{i\geq 0} x_{j+i}p^i$$

and

$$s_p\Big(\left\lfloor \frac{x}{p^l} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{x}{p^j} \right\rfloor\Big) = \sum_{i\geq 0} x_{l+i} - \sum_{i\geq 0} x_{j+i} = -\sum_{i=j}^{l-1} x_j.$$

$\square$

We want to apply this lemma as follows: We transform with assertion (ii) and (iv)

$$\begin{aligned}\left\lfloor \frac{\alpha_0-1}{p^{i_n}} \right\rfloor - \left\lfloor \frac{\alpha_0-1}{p^{i_{n-1}}} \right\rfloor &= -(p-1)\sum_{j=i_{n-1}+1}^{i_n} \left\lfloor \frac{\alpha_0-1}{p^i} \right\rfloor + s_p\Big(\left\lfloor \frac{\alpha_0-1}{p^{i_n}} \right\rfloor\Big) - s_p\Big(\left\lfloor \frac{\alpha_0-1}{p^{i_{n-1}}} \right\rfloor\Big) \\ &= -\sum_{j=i_{n-1}+1}^{i_n} \Big((p-1)\left\lfloor \frac{\alpha_0-1}{p^i} \right\rfloor + (\alpha_0-1)_{j-1}\Big),\end{aligned}$$

where $\sum_{j\geq 0}(\alpha_0-1)_j p^j$ is the p-adic expansion of $\alpha_0 - 1$. This altogether gives us

$$x_n = \sum_{j=i_{n-1}+1}^{i_n} \Big((p-1)\Big(\alpha_j - np^{e-j} - \left\lfloor \frac{\alpha_0-1}{p^j} \right\rfloor\Big) - (\alpha_0-1)_{j-1}\Big) + p^{e-i_{n-1}}.$$

Again using $(p-1)\sum_{j=i_{n-1}+1}^{i_n} p^{e-j} = p^{e-i_{n-1}} - p^{e-i_n}$, we obtain for the difference

$$\begin{aligned}x_n - x_{n+1} = &\sum_{j=i_{n-1}+1}^{i_n} \Big((p-1)\Big(\alpha_j - (n-1)p^{e-j} - \left\lfloor \frac{\alpha_0-1}{p^j} \right\rfloor\Big) - (\alpha_0-1)_{j-1}\Big) \\ &- \sum_{j=i_n+1}^{i_{n+1}} \Big((p-1)\Big(\alpha_j - (n+1)p^{e-j} - \left\lfloor \frac{\alpha_0-1}{p^j} \right\rfloor\Big) - (\alpha_0-1)_{j-1}\Big).\end{aligned}$$

In fact, the terms in the first sum are non-negative: As $0 \leq a_j \leq p-1$ by definition of the p-adic expansion, it suffices to see that

$$\begin{aligned}\alpha_j - (n-1)p^{e-j} - \left\lfloor \frac{\alpha_0-1}{p^j} \right\rfloor &= \left\lfloor \frac{n_j+a}{\nu p^j} - \frac{(n-1)m}{\nu p^j} - \frac{a}{\nu p^j} \right\rfloor \\ &= \left\lfloor \frac{n_j-(n-1)m}{\nu p^j} \right\rfloor \geq 0.\end{aligned}$$

Note that we again used the identity $\lfloor a\nu^{-1} \rfloor = \alpha_0 - 1$. By assumption, $n_j \geq (n-1)m$ and if equality holds, then $\nu p^j \mid n_j$. But then the defining equation of α_j shows that $\nu p^j \mid a$, and $(\alpha_0 - 1)_{j-1} = \lfloor a\nu^{-1} \rfloor_{j-1} = 0$. Similarly, one shows that

$$(p-1)\Big(\alpha_j - (n+1)p^{e-j} - \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor\Big) - (\alpha_0 - 1)_{j-1} \leq 0$$

and that we should rephrase $x_n - x_{n+1}$ as

$$\begin{aligned} x_n - x_{n+1} = & \sum_{j=i_{n-1}+1}^{i_n} \Big((p-1)\Big(\alpha_j - (n-1)p^{e-j} - \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor\Big) - (\alpha_0 - 1)_{j-1}\Big) \\ & + \sum_{j=i_n+1}^{i_{n+1}} \Big((p-1)\Big((n+1)p^{e-j} + \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor - \alpha_j\Big) + (\alpha_0 - 1)_{j-1}\Big), \end{aligned}$$

where now all summands are non-negative. So how does this apply to give a "nice" formula for the torsion structure? Do Euclidean division on the n_i, i.e. write $n_i = \beta_i m + \gamma_i$, where $0 \leq \gamma_i < m$ holds. Note that the inequality $i_{n-1} < i_n$ means that we have a jumping number $(n-1)m \leq i_n < nm$ and therefore

$$\beta_{i_n} = \ldots = \beta_{i_{n-1}+1} = n - 1.$$

So writing shortly $x_n - x_{n+1} = \sum_{j=i_{n-1}+1}^{i_n} y_j + \sum_{j=i_n+1}^{i_{n+1}} z_j$ with all y_j, z_j non-negative as above, we see

$$\begin{aligned} (R/\mathfrak{m}^n)^{\oplus x_n - x_{n+1}} &= (R/\mathfrak{m}^n)^{\oplus \sum_{j=i_{n-1}+1}^{i_n} y_j} \oplus (R/\mathfrak{m}^n)^{\oplus \sum_{j=i_n+1}^{i_{n+1}} z_j} \\ &= \bigoplus_{j=i_{n-1}+1}^{i_n} (R/\mathfrak{m}^{\beta_j+1})^{\oplus y_j} \oplus \bigoplus_{j=i_n+1}^{i_{n+1}} (R/\mathfrak{m}^{\beta_j})^{\oplus z_j}. \end{aligned}$$

In particular, we have

$$\begin{aligned} & (R/\mathfrak{m}^n)^{\oplus x_n - x_{n+1}} \oplus (R/\mathfrak{m}^{n+1})^{\oplus x_{n+1} - x_{n+2}} \\ = & \bigoplus_{j=i_{n-1}+1}^{i_n} (R/\mathfrak{m}^{\beta_j+1})^{\oplus y_j} \oplus \bigoplus_{j=i_n+1}^{i_{n+1}} (R/\mathfrak{m}^{\beta_j})^{\oplus z_j} \oplus \\ & \quad \bigoplus_{j=i_n+1}^{i_{n+1}} (R/\mathfrak{m}^{\beta_j+1})^{\oplus y_j} \oplus \bigoplus_{j=i_{n+1}+1}^{i_{n+2}} (R/\mathfrak{m}^{\beta_j})^{\oplus z_j} \\ = & \bigoplus_{j=i_n+1}^{i_{n+1}} \big((R/\mathfrak{m}^{\beta_j+1})^{\oplus y_j} \oplus (R/\mathfrak{m}^{\beta_j})^{\oplus z_j}\big) \oplus \text{ rest}. \end{aligned}$$

Now the jumping numbers between $i_n + 1$ and i_{n+1} do not appear in any other formula $x_l - x_{l+1}$ for l unequal to n and $n + 1$, so rearranging the direct sum yields

Theorem 2.11. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration and $m = \nu p^e$ the multiplicity of the closed fiber. For $j = 1, \dots, e$ define*

$$y_j = (p-1)\big(\alpha_j - \beta_j p^{e-j} - \big\lfloor \frac{\alpha_0 - 1}{p^j} \big\rfloor\big) - (\alpha_0 - 1)_{j-1},$$
$$z_j = (p-1)\Big((\beta_j + 1)p^{e-j} + \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor - \alpha_j\Big) + (\alpha_0 - 1)_{j-1}$$

as above. Then $R^1 f_* \mathscr{O}_X = R \oplus \mathscr{T}$, with torsion part $\mathscr{T}$ given by

$$\mathscr{T} = \bigoplus_{j=1}^{e} \big((R/\mathfrak{m}^{\beta_j+1})^{\oplus y_j} \oplus (R/\mathfrak{m}^{\beta_j})^{\oplus z_j}\big).$$

Note that we might use $R/\mathfrak{m}^0 = 0$ in the formula.

Remark 2.12. We can again calculate $l(\mathscr{T})$ by above formula: We therefore note that $y_j + z_j = (p-1)p^{e-j}$ and hence

$$\begin{aligned} l(\mathscr{T}) &= \sum_{j=1}^{e} y_j(\beta_j + 1) + \beta_j z_j \\ &= \sum_{j=1}^{e} (p-1)\beta_j p^{e-j} + (p-1)\Big(\alpha_j - \beta_j p^{e-j} - \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor\Big) - (\alpha_0 - 1)_{j-1} \\ &= (p-1)\sum_{j=1}^{e} \alpha_j - (p-1)\sum_{j=1}^{e} \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor - \sum_{j=1}^{e} (\alpha_0 - 1)_{j-1}. \end{aligned}$$

Now the equations

$$\sum_{j=1}^{e} (\alpha_0 - 1)_{j-1} = s_p((\alpha_0 - 1)),$$
$$(p-1)\sum_{j=1}^{e} \Big\lfloor \frac{\alpha_0 - 1}{p^j} \Big\rfloor = (\alpha_0 - 1) - s_p(\alpha_0 - 1)$$

are true by Lemma 2.10. Putting these equalities into the equation for $l(\mathscr{T})$ gives the known formula in Corollary 2.8.

Remark 2.13. Theorem 2.11 becomes much simpler in the case $e = 1$, i.e. the multiplicity of the closed fiber is $m = \nu p$: We then have $\lfloor (\alpha_0 - 1)p^{-1} \rfloor = 0$ because the formula $\nu\alpha_0 = 1 + a \le m = \nu p$ implies $\alpha_0 \le p$. Furthermore, $(\alpha_0 - 1)_0 = \alpha_0 - 1$ holds. If $a = 0$, then $n_1 = \alpha_1 m_1$ and hence $\beta_1 = \alpha_1$. If $a > 0$, then $n_1 = \alpha_1 \nu p - a = (\alpha_1 - 1)\nu p + (\nu p - a)$ and $\beta_1 = \alpha_1 - 1$. Calculating y_1 and z_1 in both cases, one obtains

$$\mathscr{T} = (R/\mathfrak{m}^{\alpha_1})^{\oplus p - \alpha_0} \oplus (R/\mathfrak{m}^{\alpha_1 - 1})^{\oplus \alpha_0 - 1}.$$

Example 2.14. We resume Example 1.2 resp. 1.10 to compute the torsion part of the elliptic fibration constructed there. We therefore note that the restriction of the quotient morphism $q\colon S\times_k E \to X$ to $\{\infty_1\}\times_k E$ yields the étale quotient morphism $E \to E/G = F$. Identifying G with the constant group scheme $\mathbb{Z}/p\mathbb{Z}$, we obtain the exact sequence of group schemes

$$0 \longrightarrow G \longrightarrow E \longrightarrow E/G \longrightarrow 0.$$

Dualizing and taking the rational points gives an exact sequence

$$0 \longrightarrow G^\vee(k) \longrightarrow \mathrm{Pic}^0(F) \longrightarrow \mathrm{Pic}^0(E).$$

As G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, the dual is given by μ_p, the kernel of the Frobenius $\mathbb{G}_m \to \mathbb{G}_m$. Hence $G^\vee(k)$ consists of the p-th roots of unity of k and is thus trivial. This shows that $\mathrm{Pic}^0(F) \to \mathrm{Pic}^0(E)$ is injective. As $q^*\mathscr{O}_X(F)$ is trivial, $q^*\mathscr{O}_X(F)|_E$ is also trivial and $\mathscr{O}_F(F)$ is mapped to $\mathscr{O}_E$. The injectivity shows that $\mathscr{O}_F(F)$ was already trivial, hence $\nu = \mathrm{ord}(\mathscr{O}_F(F)) = 1$.

This gives $\alpha_0 = 1 + a = p - b$. Now from the formula on the length of the torsion, Corollary 2.8, we obtain

$$m - d = l(\mathscr{T}) = 1 - \alpha_0 + (p-1)\alpha_1.$$

Solving the equation for α_1 gives $\alpha_1 = d + 1$. Using Remark 2.13, we obtain

$$\mathscr{T} = (R/\mathfrak{m}^{d+1})^{\oplus b} \oplus (R/\mathfrak{m}^d)^{\oplus p-b-1}$$

for the torsion in $R^1 f_*\mathscr{O}_X$. Here, R is the discrete valuation ring $\mathscr{O}_{\mathbb{P}^1_k,\infty}$ or its completion as used in Theorem 2.11. We will see in Example 5.24 that this torsion group coincides with the group $H^1(G, R')$, where R' is the normalization of $\widehat{\mathscr{O}}_{\mathbb{P}^1_k,\infty}$ in $K' = K[x]/(x^p - x - t^m)$.

As in [31], Example 8.4, we can modify our previous example as follows:

Example 2.15. Like in the previous example, let $K = k(t)$ be the function field of $\mathbb{P}^1_k$ with k algebraically closed and consider the Artin–Schreier extension $K' = K[x]/(x^p - x - t^{nm})$, where $p > 0$ does not divide nm. Let S be the curve corresponding to this field extension. We already saw that it had genus $g(S) = 2^{-1}(p-1)(nm-1)$ and that $\sigma(x) = x + 1$ is an automorphism generating the Galois group $\mathbb{Z}/p\mathbb{Z}$ of K' over K. Denoting by ζ_n a primitive n-th root of unity, we obtain another field automorphism given by $\tau(t) = \zeta_n t$. Note that this is not defined over K. They both extend to a group action of $G = \mathbb{Z}/p\mathbb{Z} \times \mu_n$ on S, and its quotient is again $\mathbb{P}^1_k$. This can be seen from $S/G = (S/(\mathbb{Z}/p\mathbb{Z}))/\mu_n \simeq \mathbb{P}^1_k/\mu_n \simeq \mathbb{P}^1_k$, where the first quotient is Example 1.2 and the second is immediate from the Hurwitz Formula, for the quotient is an integral normal curve and its genus must be less than or equal to the genus of $\mathbb{P}^1_k$, hence zero. Choosing an ordinary elliptic curve E over k with pn-torsion point $P \in E(k)$, we obtain an action of G on $S \times_k E$ via $\sigma(t, x, Q) = (t, x+1, Q+nP)$ and $\tau(t, x, Q) = (\zeta_n t, x, Q + pP)$. Denote by X the quotient of $S \times_k E$ by G. As G acts freely on closed points, it is again étale and X is regular. Like in Example 1.2, we see that the induced morphism $f\colon X \to S$ is an elliptic fibration, now with multiple fibers $X_0 = nE_0$ over the origin and $X_\infty = npE_\infty$ over the point at infinity. The fiber over the origin must be tame, as n is coprime to p. Again, we consider G as the kernel of the group scheme morphism $E \to E/G$. The choice of ζ_n gives an isomorphism $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. Thus, we have $G^\vee(k) = \mu_n^\vee(k) = \mathbb{Z}/n\mathbb{Z}$, from which we deduce that the order ν of $\mathscr{O}_{E_\infty}(E_\infty)$ is equal to n, so X_∞ is again a wild fiber. Note that we have now

$$\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(-2 + l)) \otimes \mathscr{O}_X((n-1)E_0) \otimes \mathscr{O}_X(aE_\infty),$$

from which we obtain, together with the isomorphisms $q^*\omega_X = \omega_{S \times_k E} = \operatorname{pr}^* \omega_S$ like in Example 1.10, the formula

$$(-2 + l)np + p(n-1) + a = 2g(S) - 2.$$

We set again $m = dp + b$ for $d \geq 0$ and $1 \leq b \leq p - 1$, yielding

$$\begin{aligned} l &= m - d, \\ a &= pn - bn - 1. \end{aligned}$$

From $\alpha_0 \nu = 1 + a$, we obtain $\alpha_0 = p - b$. Using Corollary 2.8, we get

$$m - d = l = 1 - \alpha_0 + (p-1)\alpha_1,$$

yielding $\alpha_1 = d + 1$. Hence,

$$\mathscr{T} = (R/\mathfrak{m}^{d+1})^{\oplus b} \oplus (R/\mathfrak{m}^{d})^{\oplus p-b-1}$$

as in the previous example.

Chapter 3

Regular models of torsors as quotients

Let X_K be a torsor under an abelian variety A_K, both defined over the function field K of a Dedekind scheme S. Take a finite Galois extension K' over K with Galois group G, so that $X_K(K')$ is non-empty. The normalization S' of S in K' carries a natural action of G induced by the action on K'. As K'/K is Galois, the quotient will be again S. We then would like to obtain an abelian fibration $f\colon X \to S$ with generic fiber X_K as a quotient of an abelian fibration $f'\colon A' \to S'$ with generic fiber $A_{K'}$, together with an action of G on A' such that f' is G-equivariant. In Section 3.1, we make this construction for the generic fiber. To generalize this approach in Section 3.3, we introduce the notion of models of X_K in Section 3.2 and discuss results taken from the literature we want to apply in Section 3.3 in the situation that S is the spectrum of a complete discrete valuation ring.

3.1 Torsors of abelian varieties

Definition 3.1. Let A_K be an abelian variety over a field K with the A_K-action given by translation. An *A_K-torsor* over K (also called *torsor under A_K* or *principal homogeneous space of A_K*) is a separated scheme X_K of finite type over K equipped with an A_K-action such that there exists an $A_{K^{\mathrm{sep}}}$-equivariant isomorphism $X_{K^{\mathrm{sep}}} \to A_{K^{\mathrm{sep}}}$. We call an A_K-torsor X_K *trivial* if there exists an A_K-equivariant isomorphism to A_K.

If we omit all scheme-theoretic notions in the definition, we get a set-theoretical definition of torsor. In particular, if X_K is an A_K-torsor over K, then $X_K(K^{\mathrm{sep}})$ is an $A_K(K^{\mathrm{sep}})$-torsor, because

$$A_K(K^{\mathrm{sep}}) = A_{K^{\mathrm{sep}}}(K^{\mathrm{sep}}) \simeq X_{K^{\mathrm{sep}}}(K^{\mathrm{sep}}) = X_K(K^{\mathrm{sep}})$$

holds. As $X_{K'} \simeq A_{K'}$ is projective, X_K also is by [15], Proposition 14.55. It is connected and smooth over K by descent, cf. [20], Proposition (17.7.1).

In general, we will call any field K' over K such that there exists an $A_{K'}$-equivariant isomorphism $X_{K'} \to A_{K'}$ a *splitting field* and say that X_K *splits over K'*. We can always find a splitting field finite over K:

Lemma 3.2. *Let X_K be an A_K-torsor. Then X_K is a trivial torsor if and only if $X_K(K)$ is non-empty.*

Proof. If X_K is a trivial torsor, it is isomorphic to A_K, which by definition has a rational point. So choose a point $P \in X_K(K)$. Restricting the group action to the point P yields an A_K-equivariant morphism $\phi\colon A_K = A_K \times_K \operatorname{Spec}(K) \to X_K$. As X_K is an A_K-torsor, we obtain an $A_{K^{\text{sep}}}$-equivariant isomorphism $\psi\colon X_{K^{\text{sep}}} \to A_{K^{\text{sep}}}$. Hence, the composition $\psi \circ \phi_{K^{\text{sep}}}\colon A_{K^{\text{sep}}} \to A_{K^{\text{sep}}}$ is $A_{K^{\text{sep}}}$-equivariant and thus a translation, in particular an isomorphism. Therefore, $\phi_{K^{\text{sep}}}$ is an isomorphism. By descent (cf. [15], Proposition 14.51), ϕ is an isomorphism. $\square$

To prove the claim that torsors under abelian varieties are - up to isomorphism - exactly the schemes that become isomorphic to an abelian variety over a separable closure of the ground field, we need the notion of *Albanese variety* and *Albanese torsor*.

Recall that the Albanese variety of a scheme X_K over K, together with a distinguished closed point $x_0 \in X$, is an abelian variety $\mathrm{Alb}^0_{X_K}$ over K together with a morphism $f\colon X_K \to \mathrm{Alb}^0_{X_K}$ mapping x_0 to 0 and satisfying the following universal property: Every morphism f from X_K to an abelian variety A_K with $f(x_0) = 0$ factors uniquely through f followed by an homomorphism of abelian varieties $\mathrm{Alb}^0_{X_K} \to A_K$. If X_K is proper over K, then the Picard scheme Pic_{X_K} exists (cf. [34], Corollary 4.8.13) and the Albanese variety can be identified with the dual of $P = (\mathrm{Pic}^0_{X_K})_{\text{red}}$, i.e. $\mathrm{Alb}^0_{X_K} = \mathrm{Pic}^0_{P/K}$ (cf. [22], n°236, Théorème 3.3). Furthermore, by loc. cit., the Albanese torsor is a torsor $\mathrm{Alb}^1_{X_K}$ under $\mathrm{Alb}^0_{X_K}$, together with a morphism $X_K \to \mathrm{Alb}^1_{X_K}$ universal in the sense that every morphism from X_K to a torsor T_K under A_K factors through $\mathrm{Alb}^1_{X_K}$, i.e. there are morphisms $\mathrm{Alb}^1_{X_K} \to T_K$ and $\mathrm{Alb}^0_{X_K} \to A_K$ such that the diagram

$$\begin{tikzcd} \mathrm{Alb}^0_{X_K} \times \mathrm{Alb}^1_{X_K} \arrow[r] \arrow[d] & \mathrm{Alb}^1_{X_K} \arrow[d] \\ A_K \times T_K \arrow[r] & T_K \end{tikzcd}$$

commutes. In this diagram, the horizontal morphisms are given by the group actions.

Note that $\mathrm{Alb}^0_{X_K}$ and $\mathrm{Alb}^1_{X_K}$ are the degree zero resp. one components of a more general *Albanese scheme* (cf. [61]) and that the construction of $\mathrm{Alb}^0_{X_K}$ and $\mathrm{Alb}^1_{X_K}$ as well as $X_K \to \mathrm{Alb}^1_{X_K}$ commutes with base change. In particular, if X_K is an A_K-torsor, we get a factorization of the identity on X_K

$$\begin{tikzcd} X_K \arrow[r, "\alpha"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Alb}^1_{X_K} \arrow[d, "\phi"] \\ & X_K \end{tikzcd}$$

and a homomorphism of abelian varieties $\psi\colon \mathrm{Alb}^0_{X_K/K} \to A_K$ such that

$$\begin{tikzcd}
\mathrm{Alb}^0_{X_K} \times \mathrm{Alb}^1_{X_K} \arrow[r] \arrow[d, "{(\psi,\phi)}"'] & \mathrm{Alb}^1_{X_K} \arrow[d, "\phi"] \\
A_K \times X_K \arrow[r] & X_K
\end{tikzcd} \tag{3.1}$$

commutes. The factorization $\mathrm{id}_{X_K} = \phi \circ \alpha$ implies that α is a monomorphism. Moreover, α is proper and thus a closed embedding, cf. [15], Corollary 12.92. As X_K and $\mathrm{Alb}^1_{X_K}$ are integral of the same dimension, α and therefore ϕ must be isomorphisms.

Using that the construction of the Albanese commutes with base change, we consider the diagram (3.1) over the separable closure of K. Choosing a rational point in $\mathrm{Alb}^1_{X_{K^{\mathrm{sep}}}}$ and its image in $X_{K^{\mathrm{sep}}}(K^{\mathrm{sep}})$ under $\psi_{K^{\mathrm{sep}}}$ and restricting the group action to this point, we obtain from diagram (3.1) the commutative square

$$\begin{tikzcd}
\mathrm{Alb}^0_{X_{K^{\mathrm{sep}}}} \arrow[r, "\sim"] \arrow[d, "\psi_{K^{\mathrm{sep}}}"'] & \mathrm{Alb}^1_{X_{K^{\mathrm{sep}}}} \arrow[d, "\phi_{K^{\mathrm{sep}}}"] \\
A_{K^{\mathrm{sep}}} \arrow[r, "\sim"] & X_{K^{\mathrm{sep}}}.
\end{tikzcd}$$

Thus, $\psi_{K^{\mathrm{sep}}}$ is an isomorphism and therefore ψ too by descent. We conclude:

Proposition 3.3. *Let X_K be a scheme over K. Then the following holds:*

(i) The torsor structure is unique, i.e. if X_K is a torsor under an abelian variety A_K, then the canonical morphisms $\mathrm{Alb}^1_{X_K} \to X_K$ and $\mathrm{Alb}^0_{X_K} \to A_K$ induced by the identity on X_K are isomorphism compatible with the torsor structures.

(ii) If $X_{K^{\mathrm{sep}}}$ is isomorphic to a torsor under an abelian variety, the canonical morphism $X_K \to \mathrm{Alb}^1_{X_K}$ is an isomorphism. In particular, X_K is canonically equipped with a torsor structure under $\mathrm{Alb}^0_{X_K}$.

Proof. The first assertion was already proven in the discussion before. So for the second assertion, assume that $X_{K^{\mathrm{sep}}}$ is a torsor under $A_{K^{\mathrm{sep}}}$. Then $X_{K^{\mathrm{sep}}} \to \mathrm{Alb}^1_{X_{K^{\mathrm{sep}}}}$ is an isomorphism by statement (i) and descends to the isomorphism $X_K \to \mathrm{Alb}^1_{X_K}$. □

Torsors can be classified by group cohomology up to isomorphism (for group cohomology, see also Section 5.1): Given a torsor X_K over K under A_K that becomes isomorphic to A_K over a Galois extension K' over K with Galois group G, we choose an isomorphism $\phi\colon A_{K'} \to X_{K'}$. Every field automorphism $\sigma \in G$ gives - by abuse of notation - automorphisms $\sigma\colon X_{K'} \to X_{K'}$ and $\sigma\colon A_{K'} \to A_{K'}$ induced by base change. We therefore look at the (in general non-commutative) diagram

$$\begin{tikzcd} A_{K'} \arrow[r, "\phi"] \arrow[d, "\sigma"'] & X_{K'} \arrow[d, "\sigma"] \\ A_{K'} \arrow[r, "\phi"'] & X_{K'} \end{tikzcd}$$

and "measure" the defect of its commutativity by $\xi_\sigma = \phi^{-1}\sigma\phi\sigma^{-1} \in \mathrm{Aut}_{K'}(A_{K'})$. Now $\mathrm{Aut}_{K'}(A_{K'})$ can be considered as an G-module via

$$G \times \mathrm{Aut}_{K'}(A_{K'}) \longrightarrow \mathrm{Aut}_{K'}(A_{K'}), \quad (\sigma, \psi) \longmapsto {}^\sigma\psi = \sigma \circ \psi \circ \sigma^{-1},$$

and the collection $\xi = (\xi_\sigma)_{\sigma \in G}$ gives a cocycle:

$$\xi_\sigma\, {}^\sigma\xi_\tau = (\phi^{-1}\sigma\phi\sigma^{-1})\sigma(\phi^{-1}\tau\phi\tau^{-1})\sigma^{-1} = \phi^{-1}\sigma\tau\phi\tau^{-1}\sigma^{-1} = \xi_{\sigma\tau}.$$

This cocycle depends on the chosen isomorphism ϕ, but if $\psi \colon A_{K'} \to X_{K'}$ is another isomorphism, the cocycles defined by them are cohomologous: Using the automorphism $\phi^{-1}\psi \in \mathrm{Aut}_{K'}(A_{K'})$, we obtain

$$\psi^{-1}\sigma\psi\sigma^{-1} = (\phi^{-1}\psi)^{-1}(\phi^{-1}\sigma\phi\sigma^{-1})\, {}^\sigma(\phi^{-1}\psi).$$

Furthermore, if $\theta \colon Y_K \to X_K$ is an isomorphism, Y_K and X_K give cohomologous cocycles: Let $\theta_{K'} \colon Y_{K'} \to X_{K'}$ be the isomorphism induced by base change and $\psi \colon A_{K'} \to Y_{K'}$ be any isomorphism. Set $\phi = \theta_{K'} \circ \psi$. Then the right square in the diagram

$$\begin{tikzcd} A_{K'} \arrow[r, "\psi"] \arrow[d, "\sigma"] \arrow[rr, bend left, "\phi"] & Y_{K'} \arrow[r, "\theta_{K'}"] \arrow[d, "\sigma"] & X_{K'} \arrow[d, "\sigma"] \\ A_{K'} \arrow[r, "\psi"] \arrow[rr, bend right, "\phi"'] & Y_{K'} \arrow[r, "\theta_{K'}"] & X_{K'} \end{tikzcd}$$

commutes. Therefore, $\theta_{K'}^{-1}\sigma\theta_{K'} = \sigma$ holds, from which we obtain

$$\phi^{-1}\sigma\phi\sigma^{-1} = \psi^{-1}\theta_{K'}^{-1}\sigma\theta_{K'}\psi\sigma^{-1} = \psi^{-1}\sigma\psi\sigma^{-1},$$

i.e. the cocycle defined by ϕ is cohomologous to the one defined by ψ.

Thus, we get a well-defined element $(\xi_\sigma)_\sigma \in H^1(G, \mathrm{Aut}_{K'}(A_{K'}))$ which depends only on the isomorphism class of X_K. So far, we have not used the group action on X_K at all! Instead, we could have used any *twist* X_K of A_K, i.e. any scheme X_K that becomes isomorphic to A_K over the separable closure of K. So if X_K is a torsor under A_K, we may choose an $A_{K'}$-equivariant isomorphism $\phi\colon A_{K'} \to X_{K'}$ as soon as X_K admits a K'-rational point by Lemma 3.2. It is completely determined by the image of the neutral element $0 \in A_{K'}(K')$. We thus see that for any point $P \in A_{K'}(K^{\mathrm{sep}})$, we have

$$\begin{aligned}
\xi_\sigma(P) &= (\phi^{-1}\sigma\phi)(\sigma^{-1}(P)) \\
&= (\phi^{-1}\sigma)(\phi(0) + \sigma^{-1}(P)) \\
&= \phi^{-1}(\sigma\phi(0) + P) \\
&= Q + P,
\end{aligned}$$

where $Q \in A_{K'}(K^{\mathrm{sep}})$ is defined as $\phi^{-1}\sigma\phi(0)$. As $\phi(0)$ and hence $\sigma\phi(0)$ are K'-rational points, Q must be K'-rational as well. We deduce that ξ_σ is the translation by Q and simply write $\xi_\sigma = Q \in A_{K'}(K')$, where we identify $A_{K'}(K')$ with the subgroup of translation automorphisms in $\mathrm{Aut}_{K'}(A_{K'})$. In fact, $A_{K'}(K')$ is invariant under the G-action and can thus be regarded as a G-submodule of $\mathrm{Aut}_{K'}(A_{K'})$. For the rest of this section, we prove

Proposition 3.4. *Denote by* $\mathrm{Tors}_{K'/K}$ *the set of all torsors that split over* K' *up to isomorphism and by* $\mathrm{Twist}_{K'/K}$ *the set of all twists that split over* K' *up to isomorphism. The following diagram is commutative with bijective horizontal arrows and injective vertical arrows:*

$$\begin{tikzcd}
\mathrm{Tors}_{K'/K} \arrow[r, "\sim"] \arrow[d, hook] & H^1(G, A_{K'}(K')) \arrow[d, hook] \\
\mathrm{Twist}_{K'/K} \arrow[r, "\sim"] & H^1(G, \mathrm{Aut}_{K'}(A_{K'})).
\end{tikzcd}$$

Proof of commutativity and injectivity. The commutativity is clear from the construction, and the injectivity of $\mathrm{Tors}_{K'/K} \to \mathrm{Twist}_{K'/K}$ follows from the uniqueness of the torsor structure up to isomorphism, see Proposition 3.3. If we show that the horizontal maps are bijective, the injectivity of $H^1(G, A_{K'}(K')) \to H^1(G, \mathrm{Aut}_{K'}(A_{K'}))$ automatically follows.

To show that the lower horizontal map is injective, let X_K and $\widetilde{X}_K$ be twists of A_K becoming isomorphic over K' via $\phi\colon A_{K'} \to X_{K'}$ and $\widetilde{\phi}\colon A_{K'} \to \widetilde{X}_{K'}$. Assume that the induced cocycles $\xi_\sigma = \phi^{-1}\sigma\phi\sigma^{-1}$ and $\widetilde{\xi}_\sigma = \widetilde{\phi}^{-1}\sigma\widetilde{\phi}\sigma^{-1}$ are cohomologous, i.e. $\xi_\sigma = \psi^{-1}\widetilde{\xi}_\sigma{}^\sigma\psi$ holds for some $\psi \in \mathrm{Aut}_{K'}(A_{K'})$ and all $\sigma \in G$. Rearranging terms gives $\sigma(\phi\psi^{-1}\widetilde{\phi}^{-1}) = (\phi\psi^{-1}\widetilde{\phi}^{-1})\sigma$, so the isomorphism $(\phi\psi^{-1}\widetilde{\phi}^{-1})\colon \widetilde{X}_{K'} \to X_{K'}$ descends and is already defined over K. That is to say: X_K and $\widetilde{X}_K$ are already isomorphic.

The torsor argument is exactly the same. Note that in this case, ϕ and $\widetilde{\phi}$ are $A_{K'}$-equivariant and ψ corresponds to translating by a $Q \in A(K')$. Therefore, $\phi\psi^{-1}\widetilde{\phi}^{-1}$ is again $A_{K'}$-equivariant. □

We now want to show the surjectivity by associating to every cocycle ξ an A_K-torsor X_K as a quotient of $A_{K'}$ under G with a twisted action: To every cocycle $\xi = (\xi_\sigma)_\sigma$, we associate the map

$$\rho_\xi\colon G \longrightarrow \mathrm{Aut}_K(A_{K'}), \quad \sigma \longmapsto \xi_\sigma\sigma,$$

where we consider ξ_σ as the translation-by-ξ_σ morphism on $A_{K'}$. This is a group homomorphism, as is easily seen:

$$\rho_\xi(\sigma\tau) = \xi_{\sigma\tau}\sigma\tau = (\xi_\sigma\sigma\xi_\tau\sigma^{-1})\sigma\tau = \xi_\sigma\sigma\xi_\tau\tau.$$

Indeed, it is injective because $\xi_\sigma\sigma = \mathrm{id}_{A_{K'}}$ applied to the zero element which is already defined over K and on which σ acts trivially, yields $0 = \xi_\sigma$, and σ must be the identity.

This defines a group action of G on $A_{K'}$, and its structure morphism $A_{K'} \to \mathrm{Spec}(K')$ is G-equivariant. As the Galois group G is the inverse limit of finite groups and the cohomology groups $H^1(G, \mathrm{Aut}_{K'}(A_{K'}))$ as well as $H^1(G, A_{K'}(K'))$ are direct limits according to Proposition 5.1, we may assume that the extension K'/K is finite. Then the quotient $q\colon A_{K'} \to A_{K'}/G = X_K$ over K exists due to $A_{K'}$ being projective and G finite. Moreover, choosing a cohomologous cocycle gives a quotient that is isomorphic:

Lemma 3.5. *Let ζ and ξ be cohomologous cocycles. Then ρ_ξ and ρ_ζ are conjugated and their induced quotients of $A_{K'}$ are isomorphic. In particular, the quotient does not depend up to isomorphism on the choice of a representative of a class in $H^1(G, \mathrm{Aut}_{K'}(A_{K'}))$.*

Proof. If the cocycle $\zeta = (\zeta_\sigma)_\sigma$ is cohomologous to $\xi = (\xi_\sigma)_\sigma$, there is some $\psi \in \mathrm{Aut}_{K'}(A_{K'})$ satisfying $\zeta_\sigma = \psi^{-1}\xi_\sigma {}^\sigma\psi$. So we see that ρ_ξ and ρ_ζ are conjugated by ψ:

$$\rho_\zeta(\sigma) = \zeta_\sigma\sigma = (\psi^{-1}\xi_\sigma {}^\sigma\psi)\sigma = \psi^{-1}\xi_\sigma\sigma\psi = \psi^{-1}\rho_\xi(\sigma)\psi.$$

Now let $q_1\colon A_{K'} \to X_1$ and $q_2\colon A_{K'} \to X_2$ be the quotients induced by ρ_ξ and ρ_ζ resp. Consider the commutative diagram

$$\begin{tikzcd}
A_{K'} \arrow[r, "\psi"] \arrow[d, "q_1"] & A_{K'} \arrow[r, "\psi^{-1}"] \arrow[d, "q_2"] & A_{K'} \arrow[d, "q_1"] \\
X_1 \arrow[r, dashed] \arrow[rr, dashed, bend right, "\mathrm{id}"'] & X_2 \arrow[r, dashed] & X_1,
\end{tikzcd}$$

where the dashed arrows come from the universal property of the quotient: The map $q_2\psi$ is invariant under the action of ρ_ζ, as we have

$$(q_2\psi)(\zeta_\sigma\sigma) = (q_2\psi)(\psi^{-1}\xi_\sigma {}^\sigma\psi\ \sigma) = (q_2\xi_\sigma\sigma)\psi = q_2\psi.$$

This induces the map $X_1 \dashrightarrow X_2$. Similarly, one obtains $X_2 \dashrightarrow X_1$, and the composition must be the identity due to the uniqueness of the factorizations. $\square$

Now it remains to show that $X_K = A_{K'}/G$ really gives the class of the cocycle ξ. The quotient map $q\colon A_{K'} \to X_K$ factors over $\phi\colon A_{K'} \to X_{K'}$ by the universal property of the fiber product. We note that q is a finite morphism, and that X_K is integral and normal because $A_{K'}$ is, cf. [15], Proposition 12.27 and Example 12.48. Thus, ϕ is also finite by [15], Proposition 12.11. Furthermore, it is birational and hence, by a version of Zariski's Main Theorem (e.g. [15], Corollary 12.88) an isomorphism. In fact, if we equip $X_{K'}$ with the G-action induced by base change, we see that ϕ is G-equivariant (where $A_{K'}$ is equipped with the action via ρ_ξ and not the one obtained by base change!): Denoting by $\mathrm{pr}\colon X_{K'} \to X_K$ the canonical projection obtained by base change and using its G-invariance, we compute

$$\mathrm{pr}\,\sigma^{-1}\phi\rho_\xi(\sigma) = \mathrm{pr}\,\phi\rho_\xi(\sigma) = q\rho_\xi(\sigma) = q.$$

Note that q was ρ_ξ-invariant and that ϕ was the unique morphism satisfying $\mathrm{pr}\,\phi = q$ given by the universal property of fiber product. Hence, the uniqueness of ϕ gives the equality $\phi = \sigma^{-1}\phi\rho_\xi(\sigma)$, i.e. $\sigma\phi = \phi\rho_\xi(\sigma)$. Rearranging terms, we obtain $\xi_\sigma = \phi^{-1}\sigma\phi\sigma^{-1}$. This proves Proposition 3.4 for twists.

Remark 3.6. Assume that the Galois extension K' over K such that X_K splits is not minimal, i.e. there is a Galois extension L over K with $L \subset K'$. Denote $G = \mathrm{Gal}(K'/K)$ and $N = \mathrm{Gal}(K'/L)$. Then σ is element of N if and only if $\rho_\xi(\sigma) = \sigma$:

Let $\phi\colon A_L \to X_L$ be an isomorphism. Then the base change $\phi_{K'}$ is N-equivariant and we have

$$\phi_{K'}\sigma = \sigma\phi_{K'} = \phi_{K'}\rho_\xi(\sigma) = \phi_{K'}\xi_\sigma\sigma.$$

Thus, $\xi_\sigma = 0$ and $\rho_\xi(\sigma) = \sigma$. Conversely, given the equality $\rho_\xi(\sigma) = \sigma$, we immediately deduce $\xi_\sigma = 0$, which shows that $\sigma\phi_{K'} = \phi_{K'}\sigma$ holds. Thus, $\sigma \in N$.

To prove the surjectivity for torsors, it is not enough to apply Proposition 3.3 stating that X_K must be a torsor under $\mathrm{Alb}^0_{X_K}$, because $\mathrm{Alb}^0_{X_K}$ might be a twist of A_K. Instead, we define for above isomorphism ϕ an action $\mu\colon A_{K'} \times X_{K'} \to X_{K'}$ as the morphism making the diagram

$$\begin{array}{ccc} A_{K'} \times X_{K'} & \xrightarrow{\mu} & X_{K'} \\ {\scriptstyle(\mathrm{id},\phi^{-1})}\big\downarrow & & \big\uparrow{\scriptstyle\phi} \\ A_{K'} \times A_{K'} & \longrightarrow & A_{K'} \end{array}$$

commutative, where the lower horizontal morphism is the usual addition on the abelian variety $A_{K'}$. On T-valued points $a \in A_{K'}(T)$ and $x \in X_{K'}(T)$, this means that we have $\mu(a,x) = \phi(a + \phi^{-1}(x))$. Using the Galois action of G on $X_{K'}$ and $A_{K'}$, we have on T-valued points a and x

$$\mu\sigma(a,x) = \mu(\sigma(a),\sigma(x)) = \phi(\sigma(a) + \phi^{-1}(\sigma(x))).$$

From the identity $\sigma\phi = \phi\rho_\xi(\sigma)$, we deduce $\phi^{-1}(\sigma(x)) = \rho_\xi(\sigma)\phi^{-1}(x)$ and hence

$$\begin{aligned}\phi(\sigma(a) + \phi^{-1}(\sigma(x))) &= \phi(\sigma(a) + \rho_\xi(\sigma)\phi^{-1}(x))\\ &= \phi(\sigma(a) + \xi_\sigma + \sigma(\phi{-}1(x))\\ &= \phi(\rho_\xi(\sigma)(a + \phi^{-1}(x)))\\ &= \sigma\phi(a + \phi^{-1}(x))\\ &= \sigma\mu(a, x).\end{aligned}$$

Hence, μ is G-invariant and descends to a morphism $A_K \times X_K \to X_K$. Thus, X_K is a torsor under A_K, proving Proposition 3.4. In fact, this quotient construction is very nice:

Proposition 3.7. *If K'/K is finite separable, the quotient morphism $q\colon A_{K'} \to X_K$ is finite étale.*

Proof. As K'/K is finite separable, so is $\operatorname{Spec}(K') \to \operatorname{Spec}(K)$ and thus $\operatorname{pr}\colon X_{K'} \to X_K$ by base change. Now $q = \operatorname{pr}\phi$ with ϕ being an isomorphism, so q must be étale as well. ☐

We therefore see that the torsor under an abelian variety can be recovered as an étale quotient. In the following, we will try to construct regular abelian fibrations as quotients. More precisely, we construct some quotient that is birational to a given one. This is often enough to read off significant invariants we are interested in. In the next section, we introduce the necessary background.

3.2 Models of abelian varieties

We start with the following important notion:

Definition 3.8. Let X_K be a proper smooth connected scheme over a field K and S a Dedekind scheme with function field K. We say that a normal scheme X flat and of finite type over S together with an isomorphism $X_\eta \simeq X_K$ is a *model of X_K over S*. If X is regular, we say that the model is regular. Likewise, we say that the model is proper resp. projective if X is proper resp. projective over S.

We will usually suppress the isomorphism $X_\eta \simeq X_K$ and write $X_\eta = X_K$. By definition, every fibration is a proper model of its generic fiber. Most often, one is interested in special kinds of models. For example, in the classification of algebraic surfaces, one is interested in relatively minimal regular elliptic surfaces. One may wonder if, given any torsor X_K of an elliptic curve E_K over K, such a model X exists over S and if it is unique. The existence holds for any smooth projective curve X_K over K: One may embed X_K into some projective space $\mathbb{P}^n_K$ and take its closure in $\mathbb{P}^n_S$, then normalize and resolve singularities. Finally, one has to contract the (-1)-curves. The uniqueness only holds if the genus of

X_K is positive, cf. [45], Theorem 9.3.21. Recall that we defined a genus-g-fibration to be a regular fibered surface X over S such that the genus of X_K is g. Then we obtain:

Theorem 3.9. *Let $X_K \to \mathrm{Spec}(K)$ be a smooth projective curve of genus $g \geq 1$. Then there exists a unique relatively minimal genus-g-fibration $X \to S$ which is a model of X_K.*

In fact, a stronger version of the theorem holds, namely that $X \to S$ is a *minimal fibration*. That means that every birational map of regular fibered surfaces $Y \dashrightarrow X$ over S extends to a birational morphism. In particular, given a birational morphism of regular fibered surfaces $X \to Y$, we take its inverse on a dense open set of Y. By minimality, the inverse extends to a morphism $Y \to X$. Concatenating the morphisms gives a morphism $Y \to Y$ which is the identity on a dense open subset. By separatedness, it must be the identity and $X \to Y$ is an isomorphism. This also shows that there are no (-1)-curves on X, as otherwise one could contract them. The same argument (also see [45], Proposition 9.3.13) shows:

Proposition 3.10. *Let $X \to S$ be a minimal regular fibered surface. Then the canonical map*

$$\mathrm{Aut}_S(X) \longrightarrow \mathrm{Aut}_K(X_K)$$

is bijective.

Note that if X_K has genus 0, the following example taken from [45], Remark 9.3.23, shows that there exist relatively minimal surfaces that are not minimal:

Example 3.11. Take $X_1 = \mathbb{P}^1_S$, and blow up in a closed point over $s \in S$ to obtain another fibered surface X. In X_s, the strict transform of E of $(X_1)_s$ is an exceptional divisor. We contract E to obtain a regular fibered surface X_2. Then X_1 and X_2 are relatively minimal models obtained from X, but not isomorphic as models: The identity on the generic fiber induces a birational map $X_1 \dashrightarrow X_2$ which cannot be extended to a morphism $X_1 \to X_2$ because the generic points on the fibers $(X_1)_s$, $(X_2)_s$ induce distinct valuations in $K(X)$.

Sadly, there is no such elaborated theory in higher dimension. Hironaka claimed a proof of the existence of resolution of singularities in positive characteristic on his homepage [28] in 2017 about 50 years after he gave a proof in the characteristic zero case. The correctness of his proof is not confirmed until now.

But if we restrict to the case that X does not have to be proper over S, then there is the very fruitful notion of Néron model which we shortly introduce. After that, we discuss some existence results of regular proper models that have such Néron models as dense open subsets. The most important reference on this topic is the book of Bosch, Lütkebohmert and Raynaud [9], as the original article [57] does not use the language of schemes. Néron models satisfy a universal property, which is - like most mathematical objects satisfying a universal property - more important than the actual construction. Here is the definition:

Definition 3.12. Let S be a Dedekind scheme with function field K and X_K be a smooth separated K-scheme of finite type. A *Néron model* is an S-model X which is smooth, separated, and of finite type satisfying the following universal property:

For each smooth S-scheme Y and each K-morphism $u_K\colon Y_K \to X_K$,
there is a unique S-morphism $u\colon Y \to X$ extending u_K.

This property is also called the *Néron mapping property.* There is also a more general notion (which in our sense would not necessarily be a model anymore due to the lack of being of finite type): If X_K is only smooth and separated, one calls a smooth separated scheme X over S a *Néron lft-model.* The letters "lft" stand for "locally of finite type", which holds automatically for X_K and X due to the smoothness. Néron and Néron lft-models are unique up to unique isomorphism due to the Néron mapping property. We are mostly interested in the following case (cf. [9], Theorem 1.4/3):

Theorem 3.13. *Let A_K be an abelian variety. Then there exists a Néron model A over S.*

If A is a Néron model of an abelian variety A_K, then the isomorphism

$$\mathrm{Hom}_S(Y, A) \xrightarrow{\sim} \mathrm{Hom}_K(Y_K, A_K)$$

given by the Néron mapping property extends canonically the group structure given on A_K to A: just take $Y = A \times_S A$ and the unique preimage of the multiplication map $A_K \times_K A_K \to A_K$ on the right hand side of the isomorphism. Do the same for the inversion map and the unit section. Furthermore, if $A \to S$ is an *abelian scheme*, i.e. smooth projective and each fiber is an abelian variety, then it is the Néron model of its generic fiber (cf. [9], Proposition 1.2/8). There is a close relationship between the relatively minimal regular model E of an elliptic curve E_K over S and its Néron model:

Theorem 3.14. *The Néron model of E_K is the smooth locus of E.*

For a proof, we refer to [45], Theorem 10.2.14. As in the Kodaira–Néron classification in Theorem 1.12, one is also interested in distinguishing the possible closed fibers of Néron models, but in a much coarser sense: Recall that the Néron model $A \to S$ of an abelian variety A_K is a smooth separated commutative group scheme of finite type. In particular, this holds as well for the closed fibers A_s, $s \in S$. Fixing a closed point $s \in S$ and changing the base to an algebraic closure k of $\kappa(s)$, A_k is defined over an algebraically closed field, and by Chevalley's Theorem (e.g. Theorem 9.2/1), its identity component A_k^0 is uniquely an extension of an abelian variety by a connected affine group H, which in turn splits as the product of a torus and a unipotent group. In fact, if the unipotent part is trivial, the extension is already defined over $\kappa(s)$, see [9], paragraph below Lemma 7.3/1. We make the following definition:

Definition 3.15. Let A_K be an abelian variety with Néron model A over S and $s \in S$ be a closed point. We say that

(i) A_K has *good reduction* or *abelian reduction at s* if A_s^0 is an abelian variety,

(ii) A_K has *semi-abelian reduction at s* if A_s^0 is the extension of an abelian variety by a torus.

In the literature, one often uses the term *semi-stable reduction* instead of semi-abelian reduction. Following [9] and [24], we do not use this term to avoid confusion with semi-stable models of a scheme over a field. Applying this notion to the Kodaira–Néron classification of closed fibers of relatively minimal elliptic fibrations (see Theorem 1.12), we obtain:

Theorem 3.16. *Let $f\colon E \to S$ be a relatively minimal elliptic fibration with section and $s \in S$ be a point with algebraically closed residue field. Then the following holds:*

(i) E_K has good reduction at s if and only if E_s has Kodaira type I_0.

(ii) E_K has semi-abelian reduction at s if and only if E_s has Kodaira type I_n *for $n \geq 0$.*

(iii) E_K does not have semi-abelian reduction at s if and only if E_s has Kodaira type $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$ *or* I_n^* *for $n \geq 0$.*

Note that in the case of elliptic curves, one rather says *multiplicative reduction* at s if the Kodaira type of E_s is I_n for $n \geq 1$ and *additive reduction* at s if E_K does not have semi-stable reduction at s.

Abelian varieties having semi-abelian reduction behave much better in certain ways, e.g. being semi-abelian is stable under base change. Therefore, one is interested to extend the base such that one obtains semi-abelian reduction. Grothendieck showed in [2], Théorème 3.6, Exposé IX, that this is always possible over a finite extension of the function field of S. We also cite [24], Theorem 3.3.6.4, for the case of a complete discrete valuation ring with algebraically closed residue field:

Theorem 3.17. *Let A_K be an abelian variety and S a Dedekind scheme with function field K. Then the following holds:*

(i) If A_K has semi-abelian reduction everywhere, then $A_{K'}$ has semi-abelian reduction everywhere for all finite extensions K'/K.

(ii) There is a finite Galois extension K'/K such that $A_{K'}$ has semi-abelian reduction everywhere.

(iii) If S is the spectrum of a complete discrete valuation ring with algebraically closed residue field, there exists a unique minimal extension K'/K in a separable closure of K such that $A_{K'}$ has semi-abelian reduction.

One may even give bounds on the minimal degree of the extension needed and the primes dividing it. Let g be the dimension of A_K. Then there always exists a finite extension K'/K of degree

$$N(g) = 2^{3g+\mathrm{ord}_2(g!)} \prod_{3\leq p\leq 2g+1} p^{\lfloor 2g/(p-1)\rfloor + \sum_{d\leq 2g/(p-1)} \mathrm{ord}_p(d)}$$

such that $A_{K'}$ has semi-abelian reduction, where the product runs over all prime numbers between 3 and $2g+1$, cf. [11], Theorem 6.8. In particular, semi-abelian reduction is attained over a tamely ramified field extension if $p = \mathrm{char}(k) > 2g+1$. This will be of great importance when doing reduction steps later. The question when A_K has semi-abelian reduction at certain points after base change leads to the following definiton:

Definition 3.18. Let A_K be an abelian variety with Néron model A over S and $s \in S$ be a closed point. We say that A_K has *potential abelian reduction* or *potential good reduction at* s if there exists a finite Galois extension K' over K such that $A_{K'}$ has good reduction at every point over s.

We now turn to the question if regular models containing the Néron model as in the case of elliptic curves exist and which properties they satisfy. Indeed, if S is the spectrum of complete discrete valuation ring, there are such models due to Künnemann [39], Theorem 3.5:

Theorem 3.19. *Let S be the spectrum of a complete discrete valuation ring and A_K an abelian variety with Néron model N over S having semi-abelian reduction. Then there exists a regular scheme P with projective flat structure morphism $P \to S$ such that*

(i) N is a dense, open subscheme of P,

(ii) the action of N on itself by translation extends to P,

(iii) the reduced special fiber $(A_s)_{\mathrm{red}}$ is a divisor with strict normal crossings on P. It has a stratification such that the strata are exactly the orbits for the action of N_s on P_s given in (ii).

There is also a generalization of Künnemann's result by Rozensztajn [64] to general regular, noetherian base schemes.

3.3 Abelian fibrations as quotients

In Section 3.1, we showed that any torsor under an abelian variety can be recovered as a quotient of the abelian variety with étale quotient morphism. We now raise the question if we can extend this construction to models of torsors, at least if we choose them appropriate. Indeed, there is a way in certain cases:

For simplicity, assume throughout this section that S is the spectrum of a complete discrete valuation ring R with algebraically closed residue field k and field of fraction K. Let X_K be a torsor under an abelian variety A_K and let K'/K be a finite extension with Galois group G so that $X_K(K')$ is non-empty. Denote by S' the normalization of S in K'. By Proposition 5.10, this is again the spectrum of a complete discrete valuation ring. Let $\xi = (\xi_\sigma)_\sigma$ be a cocycle of the corresponding cohomology class of X_K. By assumption, $\xi_\sigma \in A_{K'}(K')$ for every $\sigma \in G$ and it acts on $A_{K'}$ by translation. Due to the Néron mapping property (cf. Definition 3.12), this translation extends to an automorphism of the Néron model N' of $A_{K'}$ over S'. The action of G on K' extends uniquely to an action of G because S is normal and one-dimensional, cf. [45], Corollary 4.1.17. As in Section 3.1, we denote by $\sigma\colon N' \to N'$ the automorphism induced by $\sigma\colon S' \to S'$. This gives again a group homomorphism

$$\rho_\xi\colon G \longrightarrow \mathrm{Aut}_S(N'), \quad \sigma \longmapsto \xi_\sigma\sigma$$

such that the structure morphism $N' \to S'$ is G-equivariant. Note that $\xi_\sigma\sigma$ is an automorphism over S and not over S' if it is not the identity. Now assume that A_K has good reduction. Then $A_{K'}$ has also good reduction and $A' = N'$ is a projective smooth model over S', hence an abelian scheme.

Lemma 3.20. *Let A_K have good reduction, $X = A'/G$ and let $q\colon A' \to X$ be the quotient morphism. Then X is a regular, projective model of X_K over S with irreducible closed fiber and q is flat.*

Proof. As q is a finite morphism, X is projective over S if and only if A' is (cf. [15], Proposition 13.66). Hence, $X \to S$ is projective. Moreover, as taking quotients commutes with flat base change, the generic fiber is isomorphic to X_K as we have seen in Section 3.1. As a finite, dominant morphism between regular schemes is flat (cf. [15], Corollary 14.127), we only have to show the regularity of X. We therefore extend the action of $A(K)$ on X_K to all of X in the following way:

Denote by T the image of $A(K) \hookrightarrow A'(K')$ given by base change. This gives an action of T on A' which is equivariant with respect to q: Taking some $Q \in T$, the morphism qQ is invariant under the twisted action on A' because Q commutes with ξ_σ and σ for any σ in G. That is, we have $qQ\rho_\xi(\sigma) = q\rho_\xi(\sigma)Q = qQ$. Hence, $Q\colon A' \to A'$ induces a morphism $X \to X$ (denoted again by Q) by the universal property of the quotient. The uniqueness of the morphism applied to the concatenation $(-Q) \circ Q$ shows that this gives the identity and therefore every $Q \in A(K)$ defines an automorphism. It extends the action of $A(K)$ on X_K.

To see that $A(K)$ acts transitively on the rational points of the closed fiber A_k, we use the surjectivity of the *reduction map*

$$A(K) = A(S) \longrightarrow A_k(k),$$

where the equality comes from the Néron mapping property (cf. Definition 3.12) and the surjectivity of the reduction map from the smoothness of A (cf. [45], Corollary 6.2.13). Now observe that the rational points of A_k correspond to the closed points of A_k due to k algebraically closed. Moreover, the closed fiber of A' is isomorphic to A_k, as we have $A' \times_k \operatorname{Spec}(k) = A \times_S (S' \times_k \operatorname{Spec}(k)) = A \times_k \operatorname{Spec}(k)$. Hence, we may lift any closed point in A' to a point in T, so that T acts transitively on the closed points of A' and via q transitively on the closed points of X_k.

Now X is normal as the quotient by a finite group and hence has a closed singular locus of codimension greater or equal to 2 (cf. [15], Proposition 6.40, together with the openness of Y^{reg} according to [18], Corollaire 6.12.8, which uses R complete). Therefore, there are regular closed points in X. Observe that all closed points of a fibration lie on closed fibers, so we can shift a regular closed point of X via the action of T to every closed point of X, i.e. X is regular. □

The crucial ingredient of the proof lies in the ability to define an action on X that is transitive on closed points. We want to mimic the proof as far as possible in the case that A_K has semi-abelian reduction. Denote by N the Néron model of A_K over S and by N' the Néron model of $A_{K'}$ over S'. Then, as before, we obtain a group action

$$\rho_\xi \colon G \longrightarrow \operatorname{Aut}_S(N'), \quad \sigma \longmapsto \xi_\sigma \sigma.$$

The problem now is that N' will not be proper over S' as long as $A_{K'}$ does not have good reduction. By Theorem 3.19, we may choose a projective regular model P' of N'. Moreover, the action of N' on itself by translation extends to P'. By the Néron mapping property, we may consider ξ_σ as an element of $N'(S')$, so that it extends to an automorphism of P'. Again denoting by $\sigma \colon P' \to P'$ the automorphism induced by $\sigma \colon S' \to S'$, we obtain a group homomorphism

$$\rho_\xi \colon G \longrightarrow \operatorname{Aut}_S(P'), \quad \sigma \longmapsto \xi_\sigma \sigma.$$

Taking quotients yields again a projective model X of X_K over S. As A_K has semi-abelian reduction, the Néron mapping property induces an open immersion $N \times_S S' \to N'$ which is an isomorphism on the identity component (cf. [24], §2.1.5). Like in the preceding lemma, we denote by T the image of the K-rational points (or S-rational, as they coincide). As they are G-invariant, this gives an action of T on X. The problem to argue for the regularity of X is if this action is transitive on the closed points. At least, for a relatively minimal elliptic fibration $f\colon X \to S$ with multiplicative reduction, this construction applies due to [46], Proposition 8.3.

In the following, we are going to consider the situation where the quotient $q\colon P' \to X$ above induced by ρ_ξ is flat, and X hence regular (cf. [48], §21.D, Theorem 51).

Lemma 3.21. *Assume that the quotient morphism $q\colon P' \to X$ given by the action of G via ρ_ξ is flat. Denote by $r\colon N'(K) \to N'(k)$ the reduction map. Then q is étale if and only if $r(\xi_\sigma) \neq 0$ for all $\sigma \neq \mathrm{id}$ in G.*

Proof. By Theorem 3.19, the Néron model $N' \subset P'$ is also dense in the closed fiber of N'. Hence, the complement $Z = P' \setminus N'$ is of codimension greater or equal to two, and by Zariski–Nagata purity (cf. [1], X, Theorem 3.1), it suffices to check the étaleness on the Néron model. As being étale is equivalent to being smooth and quasi-finite (cf. [20], Corollaire 17.6.2 and Corollaire 17.10.2) and the smooth locus in P' is open (cf. [15], Proposition 6.15), it suffices to check étaleness on the closed points.

Take a closed point $a \in N'(k)$. This lifts to a point $A \in N'(K')$, i.e. $r(A) = a$. Furthermore, the reduction map is a group homomorphism, and we obtain

$$(r\rho_\xi(\sigma))(A) = (r\xi_\sigma\sigma)(A) = r(\xi_\sigma + \sigma(A)) = r(\xi_\sigma) + r(\sigma(A)) = r(\xi_\sigma)(r(A))$$

for every $\sigma \in G$, that is, the action of G on $N'_{s'}$ is given by the translations $r(\xi_\sigma) \in N'(k)$. Now if $r(\xi_\sigma) \neq 0$ for all $\sigma \neq \mathrm{id}$ in G, the action has no fixpoint and each orbit consists of $d = [K'\colon K]$ points. In particular, these are preimages of closed points $q(a) \in X$, and q is étale. Conversely, if $r(\xi_\sigma) = 0$ for some $\sigma \neq \mathrm{id}$, then the stabilizer of each closed point $a \in N'(k)$ is non-trivial, and its orbit consists of less than d points. In particular, the fiber $N'_{q(a)}$ is equal to this orbit. If q would be étale at a, then the preimage would consist of d copies of k (k algebraically closed) in contradiction to the size of the orbit. $\square$

In fact, we can factorize q into some "maximal" étale and non-étale part:

Proposition 3.22. *Assume that the quotient morphism $q\colon P' \to X$ given by the action of G via ρ_ξ is flat. Then the subset $H = \{\, \sigma \in G \mid r(\xi_\sigma) = 0 \,\} \subset G$ is a normal subgroup of G and allows a factorization*

$$\begin{tikzcd} P' \arrow[r, "q_1"] \arrow[d] \arrow[rr, bend left, "q"] & P'/H \arrow[r, "q_2"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S'/H \arrow[r] & S \end{tikzcd}$$

of q, where q_1 is not étale and q_2 is finite étale. Moreover, G/H is an abelian group, and the corresponding Galois extension does not depend on the choice of K'/K splitting X_K.

Proof. To see that H is a normal subgroup, we note $\mathrm{id} \in H$ and that for $\sigma, \tau \in H$, we have

$$r(\xi_{\sigma\tau^{-1}}) = r(\xi_\sigma + {}^\sigma\xi_{\tau^{-1}}) = r(\xi_\sigma) - r({}^{\sigma\tau^{-1}}\xi_\tau) = 0 - 0,$$

where we used $\xi_{\tau^{-1}} = -\ {}^{\tau^{-1}}\xi_\tau$ following from $0 = \xi_{\tau\tau^{-1}}$. Hence $\sigma\tau^{-1} \in H$ and H is a subgroup. For the normality, take $\tau \in G$ and $\sigma \in H$. Then

$$r(\xi_{\tau\sigma\tau^{-1}}) = r(\xi_\tau) + {}^\tau(r(\xi_\sigma) + {}^\sigma r(\xi_{\tau^{-1}})) = 0 + 0 + 0,$$

i.e. normality holds. Actually, we could have also argued that H is the kernel of $r\rho_\xi$.

We therefore get the factorization of q as stated in the diagram of the claim as follows: Restricting the action of G to H gives a non-étale morphism q_1 by Lemma 3.21. Now G acts on P'/H induced by the action of G on P': Given $\rho_\xi(\sigma)\colon P' \to P'$ and $\rho_\xi(\tau)\colon P' \to P'$ for $\sigma \in G$ and $\tau \in H$, we get $\sigma\tau = \tau'\sigma$ for some τ' in H by the normality. Hence we have $q_1\rho_\xi(\sigma\tau) = q_1\rho_\xi(\tau')\rho_\xi(\sigma) = q_1\rho_\xi(\sigma)$ by the H-invariance of q_1, i.e. $q_1\rho_\xi(\sigma)\colon P' \to P'/H$ factors through P'/H. In fact, as H acts trivial on P'/H, we get an induced action of G/H on P'/H. It is étale due to Lemma 3.21, for if we have $r(\xi_{\sigma\tau}) = 0$ for some $\tau \in H$ and $\sigma \in G$, it would follow that $r(\xi_\sigma) = 0$, that is to say, $\sigma \in H$.

To see that G/H is an abelian group, we show that each commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ actually lies in H. We therefore compute

$$\begin{aligned}
r(\xi_{\sigma\tau\sigma^{-1}\tau^{-1}}) &= r(\xi_\sigma + \ {}^\sigma(\xi_\tau + \ {}^\tau(\xi_{\sigma^{-1}} + \ {}^{\sigma^{-1}}\xi_{\tau^{-1}}))) \\
&= r(\xi_\sigma) + \ {}^{\sigma\tau}r(\xi_{\sigma^{-1}}) + \ {}^\sigma(r(\xi_\tau) + \ {}^{\sigma\tau\sigma^{-1}}r(\xi_{\tau^{-1}})) \\
&= r(\xi_\sigma) + r(\xi_{\sigma^{-1}}) + r(\xi_\tau) + r(\xi_{\tau^{-1}}) \\
&= r(\xi_\sigma) - \ {}^{\sigma^{-1}}r(\xi_\sigma) + r(\xi_\tau) - \ {}^{\tau^{-1}}r(\xi_\tau) \\
&= 0,
\end{aligned}$$

where we used that G acts trivially on $N'(k)$.

For the last assertion, take another Galois extension L over K splitting X_K. The compositum $L' = LK'$ is again a Galois extension over K. To compute the kernel of the map $r\rho_\xi\colon G \to N'_k(k)$, we first note that for σ in $\mathrm{Gal}(L'/K')$, we have $\xi_\sigma = 0$ due to Remark 3.6. Hence, $\mathrm{Gal}(L'/K')$ is contained in $\ker(r\rho_\xi)$, and $r\rho_\xi$ factors over $\mathrm{Gal}(L'/K)/\mathrm{Gal}(L'/K') = \mathrm{Gal}(K'/K)$. So we can restrict ourselves to the extension K' over K. Thus, we have identified the kernels of $r\rho_\xi$ for the extensions L'/K and K'/K. The same applies to L'/K and L/K, and we are done. ∎

Chapter 4

Base change of fibrations

For the whole chapter, we fix the following notation: S is the spectrum of a complete discrete valuation ring R with field of fractions K and algebraically closed residue field k. We denote by X_K a torsor under an abelian variety A_K over K and $X \to S$ will be a proper model of X_K. In this setting, we study the behaviour of the multiplicity of the closed fiber X_k under base change in the first section. To restrict the general case of abelian fibrations over Dedekind schemes to our setting is justified by Proposition 2.1, which says that the multiplicity is invariant under localization and completion of the base scheme. In the second section, we will use the results to understand étale covers of X. The whole chapter follows the treatises of [54], §5 and [53], §4, and is adjusted to abelian varieties.

4.1 The multiplicity of closed fibers under base change

Given a proper model $X \to S$ of X_K, we write $\sum_{i=1}^{r} m_i F_i$ for the closed fiber X_k considered as a Weil divisor and denote by $m = \gcd\{m_i\}$ the *multiplicity of the closed fiber* X_s. To relate m to an invariant only depending on the generic fiber, we define

$$\delta(X_K/K) = \gcd\{\, [\kappa(P) \colon K] \mid P \in X_K \text{ closed point}\,\}.$$

The following is a result from [14], Proposition 8.2(b):

Proposition 4.1. *Let X be a proper regular model of X_K over S. Then $m = \delta(X_K/K)$ holds. In particular, the multiplicity of the closed fiber depends only on the generic fiber and equals one if X_K has a rational point.*

We now make the following construction: Given a finite separable field extension K'/K of degree d, we let S' be the normalization of S in K'. As we will see in Proposition 5.10, S' is again the spectrum of a complete discrete valuation ring R' which is totally ramified of degree d over R, i.e. $\mathfrak{m}R' = \mathfrak{m}'^d$. Consider the following commutative diagram

$$
\begin{tikzcd}
X' \arrow[dr] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "f'"'] & & \\
& X \times_S S' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& S' \arrow[r] & S,
\end{tikzcd}
\tag{4.1}
$$

where $X' \to X \times_S S'$ is the normalization. This is possible because $X \times_S S'$ is integral: As $X \times_S S' \to X$ is flat with integral generic fiber, $X \times_S S'$ integral follows from [45], Proposition 4.3.8. For the following discussion, we need π_X finite. By [15], Theorem 12.51, S is an *excellent* scheme, i.e. it is locally noetherian and satisfies the following conditions:

(i) For all $s \in S$, every fiber of the canonical morphism $\operatorname{Spec}(\widehat{\mathscr{O}}_{S,s}) \to \operatorname{Spec}(\mathscr{O}_{S,s})$ is geometrically regular.

(ii) For every morphism $T \to S$ of finite type, the set of regular points $t \in T$ such that $\mathscr{O}_{T,t}$ is regular is open in T.

(iii) For every morphism $T \to S$ of finite type and for every pair of closed irreducible subsets $Z \subset Z' \subset T$, every maximal chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = Z'$ of closed irreducible subsets has the same length.

The class of excellent schemes is a rather big one: If Y is excellent and $Y' \to Y$ is a morphism locally of finite type, then Y' is also excellent, cf. [20], Proposition 7.8.6. Therefore, all schemes appearing in diagram 4.1 are excellent. Furthermore, the normalization of an integral scheme is finite (cf. [20], Proposition 7.8.6.) and thus both $S' \to S$ as well as $X' \to X \times_S S'$ are finite. This gives π_X finite. We remark that the property "excellent" is the right notion to study resolution of singularities, cf. [20], Proposition 7.9.5 and Remarque 7.9.6. The following result is [54], Lemma 5.1.2, generalized to the case of torsors of abelian varieties:

Lemma 4.2. *Let $X_k = mF$ be the closed fiber of $X \to S$ with multiplicity m. Then*

(i) the relations $m' \mid m$ and $m \mid dm'$ hold,

(ii) if π_X is étale, then $m = dm'$ holds,

(iii) if $X(K)$ is non-empty, then $m_i = 1$ for some $1 \leq i \leq r$ and $m = 1$.

Proof. To prove the relations, we denote as usual $X_k = mF$ and $X'_k = m'F'$. Denoting the morphism $S' \to S$ by g, we obtain $f'^*g^*\{\pi\} = \pi_X^* f^*\{\pi\}$ for a uniformizer $\pi \in R$, considered as a divisor, by the commutativity of the diagram. As g is totally ramified,

$g^*\{\pi\} = d\{\tau\}$ holds as divisors, where $\tau \in R'$ is again a uniformizer. Hence, we deduce $dm'F' = m\pi_X^* F$, and $m \mid dm'$.

Now recall (e.g. [45], Exercise 7.2.3; one needs additionally normality) that the pullback of F under π_X can be written as

$$\pi_X^* F = \sum_{x'} e_{x'/x} \operatorname{mult}_x(F) \overline{\{x'\}},$$

where the sum runs over all points of X' of codimension one such that $x = \pi_X(x')$ is of codimension one in X, the integer $\operatorname{mult}_x(F)$ is the multiplicity of the divisor F at x, the number $e_{x'/x} = \operatorname{length}_{\mathscr{O}_{X',x'}}(\mathscr{O}_{X',x'}/m_x \mathscr{O}_{X',x'})$ is the ramification degree and $\overline{\{x'\}}$ is the prime divisor associated to x'. If π_X is étale, it is unramified and $e_{x'/x} = 1$ for all ramification degrees. Therefore, the greatest common divisor of the multiplicities of $\pi_X^* F$ is equal to one, and $\pi_X^* F = F'$. Hence, $dm'F' = mF'$ and $m = dm'$.

More generally, let $x_1', \ldots, x_r'$ be points of codimension one in X' mapping to x of codimension one in X. Then we will see the formula $d = \sum_{i=1}^r e_{x_i'/x}[\kappa(x_i) \colon \kappa(x)]$ in Proposition 5.12. Thus, the greatest common divisor of the $e_{x_i'/x}$ has to divide d. Writing $\pi_X^*(F) = \lambda F'$, the integer λ then has to divide all $e_{x'/x}$. Hence, λ divides d and from $\lambda m = dm'$, we deduce $m' \mid m$.

For the third statement, take a point $P \in X(K)$ and denote its closure in X by C. We then have

$$1 = [\kappa(P) \colon K] = \deg_k(\mathscr{O}_X(X_s)|_C) = \sum_{i=1}^r m_i \deg_k(\mathscr{O}_X(F_i)|_C).$$

Now $\deg_k(\mathscr{O}_X(F_i)|_C) = \dim_k H^0(X, \mathscr{O}_{C \cap F_i}) \geq 0$, so C intersects exactly one component F_i which has multiplicity $m_i = 1$, and therefore $m = 1$. □

4.2 The maximal field extension inducing an étale covering of a regular S-model

Consider again the commutative diagram (4.1), where X was a proper model of X_K over S. We say that the extension K'/K of function fields corresponding to the dominant morphism of Dedekind schemes $S' \to S$ *induces an étale covering of* X if the induced morphism $\pi_X \colon X' \to X$ is étale.

Lemma 4.3. *The following statements hold:*

(i) Every field extension K'/K inducing an étale covering is separable.

(ii) Let $K''/K'/K$ be a tower of fields. Then K''/K induces an étale covering τ_X if and only if K'/K and K''/K' induce étale coverings π_X and $\pi_{X'}$.

(iii) Inducing an étale covering is stable under composition, i.e. given two field extensions K_1, K_2 over K in a common separable closure K^{sep}, each inducing an étale covering, the compositum K_1K_2/K induces an étale covering.

Proof. To prove separability, consider the reformulation

$$\mathrm{Spec}(K') = \mathrm{Spec}(\mathscr{O}_{X',\eta'}) \longrightarrow \mathrm{Spec}(\mathscr{O}_{X,\eta}) = \mathrm{Spec}(K)$$

of the generic fiber of π_X. The extension must be separable by étaleness.

To see that the subextensions K'/K and K''/K' of an extension K''/K inducing an étale covering induce étale coverings, consider the base change diagram

$$\begin{tikzcd}
X'' \arrow[r, "\pi_{X'}"] \arrow[d] \arrow[rr, bend left, "\tau_X"] & X' \arrow[r, "\pi_X"] \arrow[d] & X \arrow[d, equal] \\
X \times_S S'' \arrow[r] \arrow[d] & X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\
S'' \arrow[r] & S' \arrow[r] & S,
\end{tikzcd}$$

where $S'' = \mathrm{Spec}(\mathcal{O}_{K''})$ and $S' = \mathrm{Spec}(\mathcal{O}_{K'})$ with $\mathcal{O}_{K''}$ and $\mathcal{O}_{K'}$ being the rings of integers in K'' and K'. Furthermore, the lower rectangles are cartesian. Because both the normalization $X' \to X \times_S S'$ and the base change morphism $X \times_S S' \to X$ are affine, π_X is affine. The morphism $\pi_{X'}$ is dominant and τ_X is finite, étale, surjective by assumption. We thus apply [53], Lemma 4.7, stating that both $\pi_{X'}$ and π_X are finite, étale and surjective. In particular, K'/K and K''/K' induce étale coverings. On the other hand, if π_X and $\pi_{X'}$ are étale, then τ is, proving the "only if" direction.

For the third assertion, let $X_1 \to X$ and $X_2 \to X$ be the corresponding étale morphisms to K_1 and K_2. By base change and stability under composition, $\tau\colon X_1 \times_X X_2 \to X_i \to X$ is again étale. The generic fiber of τ equals $K_1 \otimes_K K_2$. By [20], Remarque (4.2.1.4) in the Errata et Addenda, it has Krull dimension zero. Moreover, it is regular as τ is smooth surjective and X is regular, see [15], Corollary 14.58. Hence every connected component is regular and gives an étale morphism to X. We thus may take the connected component X' of $X_1 \times_X X_2$ belonging to the prime ideal that is the kernel of the canonical map $K_1 \otimes_K K_2 \to K_1K_2$. The morphism $X' \to S$ factors over $S' = \mathrm{Spec}(\mathcal{O}_{K_1K_2})$ with $\mathcal{O}_{K_1K_2}$ being the integral closure of R in K_1K_2. It remains to show that $X' \to S'$ is actually induced by K_1K_2/K:

Restricting $\tau\colon X_1 \times_X X_2 \to X$ to X' gives a morphism $X' \to X \times_S S'$ by the universal property of the fiber product. So we get a factorisation of τ by

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[rr, bend left, "\tau"] & X \times_S S' \arrow[r] & X. \end{tikzcd}$$

Now τ is a finite morphism, and hence u is finite by cancellation. As all morphisms are dominant and $K(X \times_S S') = K_1K_2 = K(X')$, u must be the normalization morphism (see [15], Proposition 12.44), i.e. τ is induced by K_1K_2/K. $\square$

Thus, we may consider the compositum M/K of all finite field extensions inducing an étale covering. We call it the *maximal field extension inducing an étale covering of X*. The next proposition on this field extension generalizes [54], Lemma 5.1.2, on the behaviour of the multiplicity under base change to our setting:

Proposition 4.4. *Let M/K be the maximal field extension inducing an étale covering of X. Then the following holds:*

(i) M/K is a finite Galois extension and induces an étale covering.

(ii) If $g\colon Y \to X$ is a proper birational morphism between regular integral S-models of X_K, the maximal field extensions inducing an étale covering of X resp. Y coincide. In particular in the case of relative curves, M/K does not depend on the choice of a regular S-model X of X_K over S.

(iii) If K'/K a finite extension such that $X(K') \neq \varnothing$ holds, then $M \subset K'$.

Proof. For the first assertion, we note that M/K is separable as the compositum of separable extensions. The normality follows from the observation that taking some embedding $\sigma\colon M \to K^{\mathrm{sep}}$ gives a field extension $\sigma(M)/K$ that again induces an étale covering of X, hence $\sigma(M) \subset M$. Thus M/K is normal and therefore Galois. It is finite by Lemma 4.2, as every separable extension inducing an étale covering reduces the multiplicity of the closed fiber. Writing M as a compositum of finitely many fields inducing étale coverings, it must itself induce an étale covering by the previous lemma.

For the second assertion, we show that the category of finite étale coverings over X - denoted by $\acute{\mathrm{E}}\mathrm{t}(X)$ - is equivalent to $\acute{\mathrm{E}}\mathrm{t}(Y)$. One considers the base change functor $X' \mapsto Y \times_X X'$, which is well-defined as étale is stable under base change. To see that it is fully faithful, we note that $\mathscr{O}_X = g_*\mathscr{O}_Y$ due to X normal and f proper, birational. So the functor $\mathscr{E} \mapsto g^*\mathscr{E}$ from the category of finite locally free $\mathscr{O}_X$-modules to finite locally free $\mathscr{O}_Y$-modules is fully faithful ($\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{E}_2)$ is again locally free and the map $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{E}_2) \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{E}_2) \otimes g_*\mathscr{O}_Y = g_*\mathcal{H}om_{\mathscr{O}_Y}(g^*\mathscr{E}_1, g^*\mathscr{E}_2)$ is an isomorphism), cf. [23], Chapter X, Lemma 3.5.

To prove essential surjectivity, it suffices to consider a connected and hence irreducible étale covering $Y' \to Y$. Then it can also be considered as the normalization of Y in $K(Y')$, cf. [15], Proposition 12.44. We want to show that $Y' \simeq Y \times_X X'$ holds for the normalization X' of X in the function field of Y'. To see that $X' \to X$ is étale, we note that the complement of the open locus $U \subset X$ such that $g^{-1}(U) \to U$ is an isomorphism is of codimension greater or equal to two by [45], Corollary 4.4.3. We know that X' and Y' coincide over U, as both are normalization morphisms. Hence, $X' \to X$ is étale over U. Applying Zariski–Nagata purity (cf. [1], X, Théorème 3.1 or Corollaire 3.3) shows that $X' \to X$ is étale everywhere.

Using the universal property of the normalization, the morphism $Y' \to X$ factors over X' and we get a commutative diagram

$$\begin{tikzcd}
Y' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y \times_X X' \arrow[r] \arrow[d] & X' \arrow[d] \\
& Y \arrow[r] & X.
\end{tikzcd}$$

The fiber product $Y \times_X X'$ is irreducible, as its irreducible components correspond to minimal prime ideals in $K(X') \otimes_{K(X)} K(Y) = K(X')$ (remember $K(Y) = K(X)$ by assumption). Moreover, it is regular as $Y \times_X X' \to Y$ is étale by base change and the base Y is regular (cf. [45], Corollary 4.3.24). The function fields of Y', of the fiber product $Y \times_X X'$ and of X' coincide, so $Y' \to Y \times_X X'$ is birational. Hence, by a form of Zariski's Main Theorem (cf. [15], Corollary 12.88), it is an open immersion. But it is also finite due to the finiteness of $Y' \to Y$ and cancellation, and thus an isomorphism.

Therefore, the categories $\acute{\mathrm{E}}\mathrm{t}(X)$ and $\acute{\mathrm{E}}\mathrm{t}(Y)$ are equivalent, and their maximal field extensions inducing étale coverings coincide. If we assume X_1 and X_2 to be regular models of a smooth projective geometrically connected curve X_K over S, one always finds a relative curve $X \to S$ lying above X_1 and X_2, i.e. morphisms $X \to X_1$ and $X \to X_2$ respecting the isomorphisms of the generic fibers (e.g. look at [54], Lemma 3.2.6).

For the last point, we may assume $K' \cap M = K$ as $K' \cap M \subset M$ induces an étale covering and we can work on with this étale covering. So let $X' \to X$ be the normalization of X in K' and $\pi_X \colon \tilde{X} \to X$ be the étale cover induced by M. We make the base change

$$\begin{tikzcd}
X' \times_X \tilde{X} \arrow[r] \arrow[d, "\pi_{X'}"'] & \tilde{X} \arrow[d, "\pi_X"] \\
X' \arrow[r] & X.
\end{tikzcd}$$

The morphism $\pi_{X'}$ is étale by base change. The scheme $X' \times_X \tilde{X}$ is irreducible with generic fiber $K(X') \otimes_{K(X)} K(\tilde{X}) = K' \otimes_K M = K'M$ and by the étaleness of $\pi_{X'}$ regular. It is induced by the extension $K'M$ over K', and applying Lemma 4.2 (*ii*) gives $1 = [K'M \colon K']$, i.e. $M \subset K'$. $\square$

Corollary 4.5. *In the situation of Proposition 3.22 where the quotient $q \colon P' \to X$, given by the group action ρ_ξ, is flat, $M = K'^{G/H}/K$ induces the étale cover $P'/H \to X$. It is the maximal field extension inducing an étale covering of X. In particular, this is an abelian extension that sits in every extension L/K splitting X_K.*

Proof. As $X_K(K')$ is not empty, the maximal field extension inducing an étale covering M/K of X_K is a subfield of K'. Furthermore, $L = K'^{G/H}$ induces an étale covering by Proposition 3.22: Consider the following commutative diagram

$$
\begin{tikzcd}
P'/H \arrow[r, dotted] \arrow[rr, dotted, bend left] \arrow[drr, "q_2"'] & \tilde{X} \arrow[r] \arrow[dr] & X \times_S \tilde{S} \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\
 & & X \arrow[r] & S,
\end{tikzcd}
$$

where $\tilde{S}$ is the integral closure of S in L and $\tilde{X}$ the normalization of $X \times_S \tilde{S}$ in L. The dotted arrow $P'/H \to X \times_S \tilde{S}$ comes from the universal property of the fiber product and $P'/H \to \tilde{X}$ from the universal property of the normalization (remember that the normality and irreducibility of P' imply these properties for the quotient P'/H). The maps $P'/H \to X$ and $\tilde{X} \to X$ are finite of the same degree, so $P'/H \to \tilde{X}$ is finite of degree 1, i.e. an isomorphism. Therefore, $P'/H \to X$ is induced by L/K and $L \subset M$.

On the other hand, the assumption $L \subsetneq M$ implies $N = \mathrm{Gal}(K'/M) \subsetneq \mathrm{Gal}(K'/L) = H$. As above, we argue that the finite morphism $P'/N \to X$ is induced by M/K. By the very definition of M, the morphism $P'/N \to X$ must be étale. But take an element $\sigma \in H$ that is not in N. Its image in G/N is not trivial and acts on P'/N. As σ fixes every closed point on the closed fiber of P', it also has to fix every closed point of the closed fiber in P'/N. Therefore the quotient morphism $P'/N \to (P'/N)/(G/N) = X$ is not étale, i.e. M/K does not induce an étale covering. Contradiction. $\square$

Corollary 4.6. *Let d be the prime-to-p part of m, the multiplicity of the closed fiber. Then $d \mid [M \colon K]$.*

Proof. By Corollary 5.13, there is - up to isomorphism - a unique extension K'/K of degree d given by $K' = K[T]/(T^d - u)$, where $u \in R$ is a uniformizer. We have to show that K'/K induces an étale cover $\pi_X \colon X' \to X$. By Zariski–Nagata purity (cf. [1], X, Théorème 3.1), it suffices to check étaleness on the points of codimension one. So let $P \in X$ be a point of codimension one and $A = \mathscr{O}_{X,P}$ its stalk, a discrete valuation ring due to X regular. The base change $A[T]/(T^d-u)$ of A by the flat morphism $R \to R' = R[T]/(T^d-u)$ is again flat. If P lies over K, the uniformizer u is a unit in A and $\operatorname{Spec}(A[T]/(T^d-u)) \to \operatorname{Spec}(A)$ is unramified because the fiber over the residue field $\kappa(A)$ of A is given by the finite separable field extension $\kappa(A)[T]/(T^d-u)$. Hence, $\operatorname{Spec}(A[T]/(T^d-u)) \to \operatorname{Spec}(A)$ is étale and $A[T]/(T^d-u)$ is already normal.

Now if P lies over k, the base change $A[T]/(T^d-u)$ is not normal anymore: As A is a discrete valuation ring, we can write $u = at^{ml}$ in A, where a is a unit, t a uniformizer in A and l a positive integer (recall that m is the greatest common divisor of the valuations of u given by codimension one points over k). Denoting the residue class of T in $A[T]/(T^d-u)$ by $\sqrt[d]{u}$, the element $\sqrt[d]{a} = \sqrt[d]{u}\, t^{-(md^{-1})l}$ is a d-th root of a and the field of fractions L' of $A[T]/(T^d-u)$ is isomorphic to both $L(\sqrt[d]{u})$ and $L(\sqrt[d]{a})$, where $L = K(X)$. In particular, $\sqrt[d]{a}$ satisfies an integral equation, but $\sqrt[d]{a}$ is not element of $A[T]/(T^d-u)$. On the other hand, consider $A[T]/(T^d-a)$. Its field of fractions is L' and the homomorphism

$$A[T]/(T^d-u) \longrightarrow A[T]/(T^d-a), \quad T \longmapsto t^{\frac{ml}{d}}T$$

is an inclusion. Like in the case where P lies over K, as a is a unit in A and p is coprime to d, the map $\operatorname{Spec}(A[T]/(T^d-a)) \to \operatorname{Spec}(A)$ is étale. In particular, $A[T]/(T^d-a)$ is a normal ring. $\square$

Chapter 5

Group cohomology of discrete valuation rings and higher ramification groups

This chapter recapitulates some group cohomology theory we are going to use. For a Galois extension K'/K with Galois group G and an abelian variety A_K, there is a canonical action of G on the K'-valued points of A_K resp. $A_{K'}$ that respects the group structure. Thus, $A_K(K')$ can be considered as a G-module and we will define cohomology groups $H^i(G, A_K(K'))$ for $i \geq 0$. The first cohomology group is especially important to us as it classifies torsors of abelian varieties becoming trivial over K'. Torsors already appeared throughout this work as generic fibers of abelian fibrations. Now assume additionally that K is equipped with a discrete valuation giving with ring of integers R. If R is complete, the normalization R' of R in K' will be again a discrete valuation ring with maximal ideal $\mathfrak{m}'$, cf. Proposition 5.10. Then G will not act only on K', but also on R' and moreover on $R'/\mathfrak{m}'^{i+1}$ for all $i \geq -1$. The elements of G acting trivially on $R'/\mathfrak{m}'^{i+1}$ give rise to a filtration $G = G_{-1} \supset G_0 \supset G_1 \supset \dots$, the *higher ramification groups*. In the wildly ramified case, we will state and prove Sen's Theorem [65] on the structure of $H^1(G, R')$ in terms of the higher ramification groups. We will later see that this group gives the torsion part of an abelian fibration in certain cases.

5.1 Profinite group cohomology

This section relies on the exposition of Serre in [67].

A topological group G is called *profinite* if it satisfies one of the following equivalent conditions (cf. [67], Proposition 0):

(i) G is the inverse limit of finite discrete groups.

(ii) G is totally disconnected, quasi-compact and Hausdorff.

Note that for a totally disconnected, quasi-compact Hausdorff group G, we have an identification $G = \varprojlim G/H$, where H runs over all open, normal subgroups of G. As G/H is finite for $H \subset G$ open and normal. This shows the easy implication $(ii) \Rightarrow (i)$. We also want to stress that in the following, one obtains the usual group cohomology for finite groups G if one equips G with the discrete topology, which makes it profinite.

Let A be an abelian group. We equip A with the discrete topology and a continuous G-action that respects the group structure of A. Those modules will be called *discrete G-modules*. Denote for each natural number n by $C^n(G,A)$ the set of all continuous maps of G^n to A (G^n equipped with the product topology) and define the coboundary map $d\colon C^n(G,A) \to C^{n+1}(G,A)$ via

$$(df)(g_1,\dots,g_{n+1}) = g_1 \cdot f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n} f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots,g_n).$$

This gives a cochain complex $C^*(G,A)$ whose cohomology groups are denoted by $H^q(G,A)$ for $q \geq 0$. They are called the *cohomology groups of G with coefficients in A*. We now list some properties given in [67], §2.2:

Proposition 5.1. *Let $(G_i)_{i\in I}$ be an inverse system of profinite groups. Assume that for every $i \in I$, we are given a discrete G_i-module A_i, and that they form a direct system $(A_i)_{i\in I}$ such that for $i \leq j$ in I, the maps $\phi_{ij}\colon G_j \to G_i$ and $\psi_{ij}\colon A_i \to A_j$ are compatible in the sense that $\psi_{ij}(\phi_{ij}(g)a) = g\psi_{ij}(a)$ holds for all $g \in G_j$ and $a \in A_i$, i.e. ψ_{ij} is a G_j-homomorphism when A_i is considered as an G_j-module via ϕ_{ij}. Set $G = \varprojlim G_i$ and $A = \varinjlim A_i$. Then one has*

$$H^q(G,A) = \varinjlim_{i\in I} H^q(G_i,A_i)$$

for each $q \geq 0$. If A is just some discrete G-module, we have

$$H^q(G,A) = \varinjlim_{U} H^q(G/U,A^U),$$

where the limit runs over all open, normal subgroups $U \subset G$ and A^U is largest subgroup of A on which U acts trivially. We also have

$$H^q(G,A) = \varinjlim_{B} H^q(G,B),$$

where the limit runs over all finitely generated sub-G-modules of A.

We will prove in Proposition 5.9 that $H^q(G,A)$ is torsion for finite groups G and $q \geq 1$. Using the property $H^q(G,A) = \varinjlim H^q(G/U,A^U)$ shows that $H^q(G,A)$ is torsion for an arbitrary profinite group G and $q \geq 1$.

Example 5.2. Let A_K be an abelian variety over a field K. Let G be the Galois group of the extension K^{sep}/K. This is a profinite group. Considering $A_K(K^{\mathrm{sep}})$ as a discrete group, the group action $m\colon G \times A_K(K^{\mathrm{sep}}) \to A_K(K^{\mathrm{sep}})$ is continuous: If P is a point in $A_K(K^{\mathrm{sep}})$, we have to show that $m^{-1}(P)$ is open in $G \times A_K(K^{\mathrm{sep}})$. As $G \times A_K(K^{\mathrm{sep}})$ is the union of the open subsets $G \times \{Q\}$, with Q running through $A_K(K^{\mathrm{sep}})$, it suffices to show that $\{(\sigma, Q) \mid \sigma(Q) = P\}$ is open in $G \times \{Q\}$. Taking a finite Galois extension L over which Q is defined, the map $m\colon G \times \{Q\} \to A_K(K^{\mathrm{sep}})$ factors over $G/N \times \{Q\}$ as $N = \mathrm{Gal}(K^{\mathrm{sep}}/L)$ acts trivially on Q. Now N is open in G and $G/N \times \{Q\}$ is a finite set with the discrete topology, so the preimage of $\{P\}$ is open in it. As the map $G \times \{Q\} \to G/N \times \{Q\}$ is continuous, the set $\{(\sigma, Q) \mid \sigma(Q) = P\}$ is open in $G \times \{Q\}$. Thus, $A_K(K^{\mathrm{sep}})$ is a discrete G-module and we may consider the cohomology group $H^1(G, A(K^{\mathrm{sep}}))$ as is done in Chapter 3.

From the explicit description of the coboundaries and the cochains, one deduces easily the following description of the first two cohomology groups:

Lemma 5.3. *The following equalities hold:*

$$H^0(G, A) = A^G$$
$$H^1(G, A) = \frac{\{\, f\colon G \to A \text{ continuous} \mid f(g_1 g_2) = f(g_1) + g_1 \cdot f(g_2) \text{ for any } g_1, g_2 \in G \,\}}{\{\, f\colon G \to A \text{ continuous} \mid f(g) = g \cdot a - a \text{ for some } a \in A \,\}}$$

In the preceding discussion, one could also forget the topology on G and do the same constructions, but the resulting cohomology groups will in general be different. Yet, if G is a finite group, they give the same cohomology groups stemming from the fact that G is discrete and thus all cocycles and coboundaries are continuous.

Example 5.4. Let G be the profinite completion $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ and let $A = \mathbb{Q}$ be equipped with the discrete topology and trivial G-action. Using Proposition 5.1, we have $H^1(G, A) = \varinjlim H^1(\mathbb{Z}/n\mathbb{Z}, A)$. As the G-action is trivial, $H^1(\mathbb{Z}/n\mathbb{Z}, A)$ can be identified with group homomorphisms $\mathbb{Z}/n\mathbb{Z} \to A$ by Lemma 5.3. Therefore, we deduce

$$H^1(G, A) = \varinjlim_n \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0,$$

as $\mathbb{Q}$ is torsion-free.

Now on the other hand, dropping the condition "continuous" in our construction of cohomology would give again Lemma 5.3, only discarding the condition "continuous" in the description. But now there are non-trivial group homomorphisms $\widehat{\mathbb{Z}} \to \mathbb{Q}$: Take the canonical inclusion $\mathbb{Z} \to \widehat{\mathbb{Z}}$ and the inclusion $\mathbb{Z} \to \mathbb{Q}$. As $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, the the homomorphism $\mathbb{Z} \to \mathbb{Q}$ extends to $\widehat{\mathbb{Z}}$. Therefore, the resulting first cohomology group is non-trivial.

There is also another description of group cohomology as the derived functor of $A \mapsto A^G$. For G profinite, we refer to [63], §6. As for the forthcoming discussion we are only interested in finite groups and to simplify the discussion, we do not care about the topology. Let G be a group and A be a G-module. Given a commutative ring R, we impose on R the trivial G-action and define $R[G]$ to be the free algebra $R[T_g, g \in G]$ over R with relations $T_{g_1}T_{g_2} = T_{g_1g_2}$. Often, A is not only a G-module, but an $R[G]$-module. We then observe that

$$H^0(G, A) = A^G = \mathrm{Hom}_{R[G]}(R, A)$$

as R-modules, where the right equality comes from the map $a \mapsto (1 \mapsto a)$ with inverse $\varphi \mapsto \varphi(1)$. One then observes that the derived functors of $\mathrm{Hom}_{R[G]}(R, \_)$ coincide with the group cohomology:

Theorem 5.5. *For any $R[G]$-module A and $q \geq 0$, we have $H^q(G, A) = \mathrm{Ext}^q_{R[G]}(R, A)$, considered as an R-module.*

This point of view has the advantage of being more flexible in terms of choosing a resolution:

Example 5.6. Let $G = \mathbb{Z}$. Then for any ring R, we have $R[G] = R[T^{\pm 1}]$. Consider the exact sequence

$$0 \longrightarrow R[T^{\pm 1}] \xrightarrow{T-1} R[T^{\pm 1}] \longrightarrow R \longrightarrow 0.$$

Applying $\mathrm{Hom}_{R[G]}(\_, A)$ for some $R[G]$-module A, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{R[T^{\pm 1}]}(R, A) \longrightarrow \mathrm{Hom}_{R[T^{\pm 1}]}(R[T^{\pm 1}], A) \xrightarrow{T-1} \mathrm{Hom}_{R[T^{\pm 1}]}(R[T^{\pm 1}], A) \longrightarrow 0.$$

Using the canonical isomorphism $\mathrm{Hom}_{R[T^{\pm 1}]}(R[T^{\pm 1}], A) = A$, we deduce

$$H^i(G, A) = \begin{cases} A^G & i = 0, \\ A/(T-1)A & i = 1, \\ 0 & i \geq 2. \end{cases}$$

We interpret $H^0(G, A)$ as the largest submodule of A with trivial G-action and $H^1(G, A)$ as the largest quotient with trivial G-action.

Example 5.7. Let G be the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$. Then we identify $S = R[G]$ with $R[T]/(T^n - 1)$ and define the maps $\phi, \psi \colon S \to S$ by $\phi(T) = 1 - T$ resp. $\psi(T) = 1 + T + \ldots + T^{n-1}$. As the concatenations $\phi \circ \psi$ and $\psi \circ \phi$ are zero maps, they give a projective periodic resolution

$$0 \longleftarrow R \longleftarrow S \xleftarrow{\phi} S \xleftarrow{\psi} S \xleftarrow{\phi} S \xleftarrow{\psi} \ldots.$$

Again applying $\mathrm{Hom}_S(\_, A)$, we obtain the cochain complex

$$0 \to \mathrm{Hom}_S(R, A) \to \mathrm{Hom}_S(S, A) \xrightarrow{\phi} \mathrm{Hom}_S(S, A) \xrightarrow{\psi} \mathrm{Hom}_S(S, A) \xrightarrow{\phi} \mathrm{Hom}_S(S, A) \xrightarrow{\psi} \dots$$

Using the identification $\mathrm{Hom}_S(S, A) = A$ and $\mathrm{Hom}_S(R, A) = A^G$ and denoting by $A[1-T]$ and $A[1 + T + \ldots + T^{n-1}]$ the kernels of ϕ and ψ, this gives

$$H^q(G, A) = \begin{cases} A^G & q = 0, \\ A[1 + T + \ldots + T^{n-1}]/(1 - T)A & q \text{ odd}, \\ A[1 - T]/(1 + T + \ldots + T^{n-1})A & q \geq 2 \text{ even}. \end{cases}$$

Note that in this context, $A[1-T]$ is the invariant ring A^G and $(1+T+\ldots+T^{n-1})\colon A \to A$ is the trace morphism $\mathrm{Tr}\colon A \to A$ which maps a to $\sum_{g\in G} g \cdot a$. We observe that for $q \geq 1$, the cohomology groups in odd dimension are all equal and the ones in even dimension are all equal. In particular, if the group acts trivially, we get $H^{2q+1}(G, A) = A[n]$, the kernel of multiplication by n, and $H^{2q+2}(G, A) = A/nA$ for $q \geq 0$. Moreover, if the group is trivial (i.e. the case $n = 1$), we obtain $H^q(G, A) = 0$ for $q \geq 1$.

We now want to elucidate some functorial properties: If we work with a profinite group, a closed subgroup $H \subset G$ is again profinite with respect to the subspace topology. It acts continuously on A and therefore gives homomorphisms $H^q(G, A) \to H^q(H, A)$, called *restriction maps*. If moreover H is normal in G, we may consider the quotient group G/H, again a profinite group with respect to the quotient topology. It does not act on A anymore, but on A^H. It induces a homomorphism $H^q(G/H, A^H) \to H^q(G, A)$, called *inflation map*. One checks that the *inflation-restriction sequence*

$$0 \longrightarrow H^1(G/H, A^H) \longrightarrow H^1(G, A) \longrightarrow H^1(H, A)^{G/H}$$

is exact, cf. [66], Chapter VII, §6. In fact, these are the first terms of the five-term exact sequence of the *Lyndon–Hochschild–Serre spectral sequence*, which can be found in [58], Chapter II, §4:

Theorem 5.8. *If $H \subset G$ is a normal and open subgroup, there is a spectral sequence $E_2^{pq} = H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A)$ which gives an exact sequence*

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H) \to H^2(G, A).$$

If $H \subset G$ is normal and open, the quotient G/H is finite and we can define the *norm map*

$$A^H \longrightarrow A^G, \quad a \longmapsto \sum_{gH\in G/H} ga.$$

This is well-defined and gives homomorphisms $H^q(H, A) \to H^q(G, A)$, called *corestrictions*. The composition with the restriction in degree zero, i.e. $A^G \to A^H \to A^G$, is nothing but multiplication with the index $[G\colon H]$. In fact this gives multiplication by $[G\colon H]$ on all cohomology groups. We get the following commutative diagram

$$\begin{tikzcd} H^q(G,A) \arrow[rr, "\text{[G : H]}"] \arrow[dr, "\text{restriction}"'] & & H^q(G,A) \\ & H^q(H,A), \arrow[ur, "\text{corestriction}"'] & \end{tikzcd}$$

from which we deduce:

Proposition 5.9. *In above situation, the following statements hold:*

(i) *The kernel of the restriction map is annihilated by $[G : H]$.*

(ii) *If $p \nmid [G : H]$, then the restriction is injective on the p-primary part of $H^q(G, A)$.*

(iii) *If G is finite of order n, then $H^q(G, A)$ is annihilated by n. If moreover multiplication by n on A is bijective, then the isomorphism $n\colon H^q(G, A) \to H^q(G, A)$ is the zero map, i.e. $H^q(G, A) = 0$ for $q \geq 1$.*

5.2 Higher ramification groups

Our goal is to study the torsion in the first higher direct image of an abelian fibration. By Proposition 2.1, this can be done locally on the base. So we may assume our base S to be the spectrum of a complete discrete valuation ring R. Working with complete discrete valuation rings has the nice property that this notion is stable under finite extensions, cf. [66], Chapter II, §2, Proposition 3:

Proposition 5.10. *Let K be a complete field with respect to some discrete valuation ν defining a discrete valuation ring $R \subset K$. Take a finite extension K'/K of degree n and let R' be the integral closure of R in K'. Then R' is a discrete valuation ring, a free module of rank n over R and K' complete with respect to the valuation induced by R'.*

Without the completeness assumption, there are in general finitely many prime ideals $\mathfrak{m}'_1, \ldots, \mathfrak{m}'_l$ in R' lying above $\mathfrak{m} \subset R$, i.e. $\mathfrak{m}'_i \cap R = \mathfrak{m}$. Note that every prime ideal is a maximal ideal due to the rings being Dedekind rings. We may thus write

$$\mathfrak{m}R' = \mathfrak{m}_1'^{e_1} \ldots \mathfrak{m}_l'^{e_l}.$$

The integer $e_i \geq 0$ is called the *ramification index of* $\mathfrak{m}'_i$ *in* K'/K. On the other hand, the canonical map $R \to R' \to R'/\mathfrak{m}'_i$ gives a finite field extension $R/\mathfrak{m} \to R'/\mathfrak{m}'_i$ of degree f_i, called the *residue degree of* $\mathfrak{m}'_i$ *in the extension* K'/K.

Example 5.11. Let $R = \mathbb{Z}$ with field of fractions $K = \mathbb{Q}$. Take the Galois extension $K' = \mathbb{Q}(i)$ over K of degree 2. Then the normalization of R in K' is $R' = \mathbb{Z}[i]$, the Gaussian integers: Every $x \in \mathbb{Q}(i)$ is of the form $x = a + bi$ and root of the polynomial $(T-(a+bi))(T-(a-bi)) = T^2 + (-2a)T + (a^2+b^2)$. Hence, if x is in the normalization, it is either in $\mathbb{Z}$ or its minimal polynomial is of the form $T^2 + (-2a)T + (a^2 + b^2)$ with integral coefficients. We conclude that $2a \in \mathbb{Z}$ and from $4(a^2 + b^2) \in \mathbb{Z}$ that $2b \in \mathbb{Z}$ as well. Moreover, $(2a)^2 + (2b)^2 \equiv 0 \bmod 4$, which is only possible if $(2a)^2 \equiv (2b)^2 \equiv 0 \bmod 4$. This gives $4a$ and $4b$ in $4\mathbb{Z}$, that is, a and b are integers.

We now study which maximal ideals lie over a given maximal ideal $(p) \subset \mathbb{Z}$ by examining the fibers of $S' = \operatorname{Spec}(R') \to \operatorname{Spec}(R) = S$: The fiber over the point corresponding to the prime p is given by the spectrum of $\mathbb{F}_p[T]/(T^2+1)$. If $p = 2$, the polynomial $T^2 + 1 = (T+1)^2$ is reducible and the fiber is a thickened point. This means that there is only one maximal ideal $\mathfrak{m}'$ lying above (2) and $2\mathbb{Z}[i] = \mathfrak{m}'^2$. If p is odd, the polynomial T^2+1 has a solution if and only if -1 is a square in $\mathbb{F}_p$, i.e. the *Legendre Symbol* $(\frac{-1}{p})$ is one. These are exactly the primes of the form $p = 4z + 1$, and for those, we have $\mathbb{F}_p[T]/(T^2+1) = \mathbb{F}_p \times \mathbb{F}_p$ by the Chinese Remainder Theorem. If p is of the form $4z + 3$, the polynomial T^2+1 is irreducible and $\mathbb{F}_p[T]/(T^2+1) = \mathbb{F}_{p^2}$. Hence, if $p \equiv 1 \bmod 4$, the fiber is given by two points which correspond to two maximal ideals over (p) of ramification index one and residue degree one. If $p \equiv 3 \bmod 4$, the fiber consists of one bigger point, which corresponds to a single maximal ideal lying above (p) of ramification index one and residue degree two.

The ramification indices and residue degrees satisfy the following proposition (cf. [66], Chapter I, §4, Proposition 10):

Proposition 5.12. *The ring $R'/\mathfrak{m}R'$ is an $R/\mathfrak{m}$-algebra of degree $n = [K' \colon K]$ isomorphic to $\prod_{i=1}^{r} R'/\mathfrak{m}_i'^{e_i}$ and the following formula holds:*

$$n = \sum_{i=1}^{r} e_i f_i.$$

In the special case of having only one prime ideal $\mathfrak{m}' \subset R'$ lying above $\mathfrak{m}$ and $f = 1$ (which by above formula is equivalent to $e = n$), we say that *K'/K is totally ramified at $\mathfrak{m}$*. The other extremal case would be that $e_i = 1$ for every $i = 1, \dots, r$ and $R'/\mathfrak{m}_i'$ is separable over $R/\mathfrak{m}$. In this case, we say that *K'/K is unramified over $\mathfrak{m}$*. Applied to Example 5.11, this means that the extension $\mathbb{Q}(i)/\mathbb{Q}$ is totally ramified at (2) and unramified over the primes p satisfying $p \equiv 1 \bmod 4$.

Corollary 5.13. *Let K be a complete discrete valuation field with algebraically closed residue field k. Then every field extension K'/K of degree n prime to $p = \operatorname{char}(k)$ is of* *the form $K' \cong K[T]/(T^n - u)$, where u is a uniformizer of R. Moreover, K'/K is a cyclic Galois extension.*

Proof. Let $u' \in R'$ be a uniformizer. As k is algebraically closed, the extension K'/K is totally ramified of degree n and there exists a unit $r \in R'^{\times}$ such that $u = ru'^n$. As the residue fields coincide, we can find an element $v \in R^{\times}$ such that its image in R' satisfies $v \equiv r \bmod \mathfrak{m}'$. Thus, replacing the uniformizer u by $v^{-1}u$, we may assume that $r \equiv 1 \bmod \mathfrak{m}'$. The polynomial $T^n - 1 \in k[T]$ has a root in k, so the lift $T^n - r \in K'[T]$ of it also admits a solution in K' by Hensel's Lemma. Thus, there is some $w \in K$ such that $w^n = r$. Replacing u' by $w^{-1}u'$, we obtain $u'^n = u$ and u' is obviously an element of degree n. For the cyclic statement, this is standard Galois theory (cf. [40], Chapter VI, Theorem 6.2). $\square$

In light of the propositions above, when working over a complete discrete valuation ring R with algebraically closed residue field $k = R/\mathfrak{m}$, we only have one prime $\mathfrak{m}' \subset R'$ lying above $\mathfrak{m} \subset R$ with $f = 1$ due to k being algebraically closed. Hence, if the extension K'/K is not trivial, we will only work with totally ramified extensions of ramification index $e = [K' \colon K]$. This is the case to keep in mind. But the following discussion holds in greater generality: Until the end of this section, if not stated otherwise, we assume R to be a complete discrete valuation ring with residue field k of characteristic p and K'/K to be a Galois extension with Galois group G. We denote by R' the integral closure of R in K' and its residue field k' to be a separable extension over k.

We now consider the group action of G on the subring $R' \subset K'$. As every element a in R' can be written as the root of a monic polynomial with coefficients in R, it will be mapped to another root of this polynomial under any automorphism of K' over K. Hence, its image will again be integral and therefore be contained in R', thus R' is closed under the action of G. Surely, the unique maximal ideal will be mapped to itself by an automorphism, and we deduce $\sigma(\mathfrak{m}'^i) = \sigma(\mathfrak{m}')^i$ for any σ in G and $i \geq 0$, inducing a well-defined action of G on $R'/\mathfrak{m}'^i$. We therefore get a filtration

$$G_i = \{\, \sigma \in G \mid \sigma \text{ operates trivially on } R'/\mathfrak{m}'^{i+1} \,\}$$

of G for $i \geq -1$. The group G_i is called the *i-th ramification group.*

Proposition 5.14. *The ramification groups give a descending filtration of $G = G_{-1}$ by normal subgroups $G \supset G_i$ with $G_i = \{\mathrm{id}_{K'}\}$ for i large enough.*

Proof. As $R'/\mathfrak{m}'^0 = 0$, the equality $G_{-1} = G$ is immediate. To show that $G_i \supset G_{i+1}$ holds, take some $\sigma \in G_{i+1}$, i.e. $\sigma(x) - x$ is in $\mathfrak{m}'^{i+2} \subset \mathfrak{m}'^{i+1}$ for any $x \in R'$. Thus, σ lies in G_i.

For the normality, take $\sigma \in G_i$ and $\tau \in G$. They give automorphisms of $R'/\mathfrak{m}'^{i+1}$ with $\sigma|_{R'/\mathfrak{m}'^{i+1}} = \mathrm{id}_{R'/\mathfrak{m}'^{i+1}}$, so in particular

$$(\tau\sigma\tau^{-1})|_{R'/\mathfrak{m}'^{i+1}} = \mathrm{id}_{R'/\mathfrak{m}'^{i+1}}\,.$$

Hence, $\tau\sigma\tau^{-1}$ is element of G_i. To see that the filtration terminates, assume that σ in G lies in all G_i, that is, $\sigma(x) - x \in \mathfrak{m}'^{i+1}$ for all $x \in R'$ and $i \geq -1$. Then $\sigma(x) - x$ lies in all intersections $\bigcap_i \mathfrak{m}'^{i+1} = 0$, i.e. σ is the identity on R'. As G contains only finitely many elements, this proves the termination of the filtration. □

We now take a look at the quotients G_i/G_{i+1}. For details, we refer to [66], Chapter IV, §2. The extension k'/k is normal by [66], Chapter I, §7, Proposition 20, and thus, by our general assumption k'/k separable, Galois. The map

$$G \longrightarrow \mathrm{Gal}(k'/k)$$

is surjective (cf. [66], Chapter I, §7, Proposition 20). The kernel, that is, the group of automorphisms acting trivial on k', is also called *inertia group* and coincides by its very definition with G_0. To get information on the other quotients, we define a decreasing filtration on the units $U = R'^{\times}$ by defining $U^0 = U$ and $U^i = 1 + \mathfrak{m}'^i$ for $i \geq 1$. We observe the following:

Proposition 5.15. *The group U^0/U^1 is canonically isomorphic to the multiplicative group $k'^{\times}$. For $i \geq 1$, the quotient U^i/U^{i+1} is canonically isomorphic to the group $\mathfrak{m}'^i/\mathfrak{m}'^{i+1}$, which is a one-dimensional k' vector space.*

Proof. The first assertion immediately follows from the surjection $U^0 \to k'^{\times}$. For the second assertion, consider the surjection $\mathfrak{m}'^i \to U^i/U^{i+1}$ mapping $x \mapsto 1 + x$. This is a group homomorphism, for $(1 + x)(1 + y) = 1 + (x + y)$ holds in the quotient. Its kernel consists of the elements of $\mathfrak{m}'^{i+1}$, proving $\mathfrak{m}'^i/\mathfrak{m}'^{i+1} = U^i/U^{i+1}$, which is a k'-vector space of dimension one. □

To establish a connection to G_i, we fix a uniformizer $u' \in R'$. We then define maps

$$G_i \longrightarrow U^i, \quad \sigma \longmapsto \frac{\sigma(u')}{u'}$$

for $i \geq 0$. These are well-defined, for we have $\sigma(u') - u' \in \mathfrak{m}'^{i+1}$ by assumption and factoring out the uniformizer, this means that $\sigma(u')(u')^{-1} - 1 \in \mathfrak{m}'^i$. This is equivalent to $\sigma(u')(u')^{-1} \in U^i$. This map induces a group homomorphism

$$\varphi\colon G_i/G_{i+1} \longrightarrow U^i/U^{i+1}, \quad \sigma \longmapsto \frac{\sigma(u')}{u'},$$

which is injective:

Proposition 5.16. *Above map φ does not depend on the choice of a uniformizer and is an injective group homomorphism.*

Proof. We first show that φ does not depend on the choice of the uniformizer u': Taking another uniformizer $t = ru'$ with $r \in U^0$, we calculate

$$\frac{\sigma(t)}{t} = \frac{\sigma(ru')}{ru'} = \frac{\sigma(u')}{u'}\frac{\sigma(r)}{r}.$$

As by the definition of $\sigma \in G_i$, we have $\sigma(r) \equiv r \mod \mathfrak{m}'^{i+1}$. This in turn yields that $\sigma(r)r^{-1}$ is equivalent to 1 mod $\mathfrak{m}'^{i+1}$, showing that $\sigma(u')(u')^{-1} = \sigma(t)t^{-1}$ in U^i/U^{i+1}.

For the homomorphism property, take σ and τ in G_i. Then

$$\frac{\sigma\tau(u')}{u'} = \frac{\sigma(u')}{u'}\frac{\sigma(\frac{\tau(u')}{u'})}{u'(u')^{-1}} = \frac{\sigma(u')}{u'}\frac{\tau(u')}{u'}\frac{\sigma(\tau(u')u'^{-1})}{\tau(u')(u')^{-1}}$$

holds, so we have to show that the last factor is in U_{i+1}. But this is immediate from the argument $\sigma(r)r^{-1} \equiv 1 \mod \mathfrak{m}'^{i+1}$, now taking $r = \tau(u')(u')^{-1} \in U^i \subset U$.

For the injectivity, let $\sigma(u')(u')^{-1}$ be trivial, that is, $\sigma(u')(u')^{-1} = 1 + a$ for some $a \in \mathfrak{m}'^{i+1}$. From this, we deduce $\sigma(u') - u' = u'a \in \mathfrak{m}^{i+2}$, i.e. $\sigma \in G_{i+1}$. Note that this must be proven for every element in R' actually, but as u' generates R' as an R-algebra, it is enough to check at a uniformizer. □

We now reap the fruits of our efforts:

Corollary 5.17. *The group G_0/G_1 is cyclic of order prime to the characteristic of the residue field. For $i \geq 1$, the groups G_i/G_{i+1} are direct products of elementary abelian p-groups. In particular, if $p = 0$, then $G_1 = \{1\}$. Otherwise, G_1 is a p-group and G_0 is the semi-direct product $G_1 \rtimes G_0/G_1$.*

Proof. By the propositions above, we may embed G_0/G_1 into $U^0/U^1 = k'^{\times}$. As G_0/G_1 is finite, it must be cyclic of order prime to the characteristic p of k'. Again, we embed G_i/G_{i+1} into $U^i/U^{i+1} \subset k'$. If $p = 0$, there is no finite additive subgroup and the quotient is trivial. If $p > 0$, every element is annihilated by p, and G_i/G_{i+1} is elementary abelian. Thus, each element in G_1 is of order power of p and G_1 is a p-group. Applying the Schur–Zassenhaus Theorem on the normal subgroup G_1, we deduce that $G_0 = G_1 \rtimes G_0/G_1$. □

5.3 Group cohomology of extensions of complete discrete valuation rings

We are now in the situation of describing the cohomology groups $H^1(G, R')$ for the integral closure R' of a complete discrete valuation ring R in the Galois extension K'/K with

Galois group G. The residue field $k = R/\mathfrak{m}$ is assumed to be algebraically closed, hence the extension K'/K is totally ramified. We start with the case that $p = \mathrm{char}(k)$ does not divide $|G|$, which means that $G_1 = \{1\}$ by Corollary 5.17. In this case, we call K'/K *tamely ramified* and the cohomology groups $H^q(G, R') = 0$ vanish for $q \geq 1$:

Proposition 5.18. *Assume that the totally ramified extension K'/K is tamely ramified. Then*

$$H^q(G, R') = 0$$

for $q \geq 1$.

Proof. By Proposition 5.9, as multiplication by $|G|$ is an isomorphism on R', the cohomology groups $H^q(G, R')$ vanish for $q \geq 1$. □

This gives us a powerful simplification of the general case: Recall that in our situation, G coincides with its inertia group G_0, so that the order of G/G_1 and p are coprime. Thus, multiplication on any R'-module with $[G\colon G_1]$ is an isomorphism, so the Lyndon–Hochschild–Serre spectral sequence $E_2^{rs} = H^r(G/G_1, H^s(G_1, R')) \Rightarrow H^{r+s}(G, R')$ (Theorem 5.8) degenerates and we immediately obtain the following proposition:

Proposition 5.19. *We have a canonical identification*

$$H^q(G, R') = H^q(G_1, R')^{G/G_1}$$

for $q \geq 0$.

In particular, as $H^q(G_1, R')$ is a torsion module, the following result from [35], Lemma 1.5, helps in taking the invariants:

Lemma 5.20. *Let K'/K be a finite Galois extension of complete discrete valuation fields of ramification index d. Then $(\mathfrak{m}'^z)^G = \mathfrak{m}^a$ holds for $z \in \mathbb{Z}$, where $a = 1 + \lfloor \frac{z-1}{d} \rfloor$.*

So we are "only" left to study the case $G = G_1$, i.e. K'/K is *wildly ramified*. Let us focus on the case where G_1 is cyclic of order p^e with generator $\sigma \in G$. Then we want to measure when an element $\sigma^l \in G$ stops being in a higher ramification group, i.e. $\sigma^l \in G_j \setminus G_{j+1}$. In other words: When does $\sigma^l(u') - u'$ lie in $\mathfrak{m}^{j+1} \setminus \mathfrak{m}^{j+2}$ for a uniformizer $u' \in R'$? Denoting by $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ the discrete valuation induced by R', this is equivalent to saying that $\nu((\sigma^l(u') - u')(u')^{-1}) = j$. More generally, we define the map

$$i\colon \mathbb{Z} \longrightarrow \mathbb{N}_{\geq 1}, \quad l \longmapsto \nu\left(\frac{\sigma^l(u') - u'}{u'}\right) = \nu(\sigma^l(u') - u') - 1.$$

Having fixed the notation, we state Sen's Theorem (cf. [65], Theorem 2):

Theorem 5.21. *In above situation, we have an isomorphism*

$$H^1(G, R') \simeq \bigoplus_{l=1}^{p^e-1} R/\mathfrak{m}^{n_l} R,$$

where $n_l = \left\lfloor \frac{l+i(l)}{p^e} \right\rfloor$.

Sketch of proof. By Example 5.7, $H^1(G, A)$ is isomorphic to $R'[\mathrm{Tr}]/(\sigma - \mathrm{id}_{R'})R'$, where $\mathrm{Tr} = \mathrm{id}_{R'} + \sigma + \cdots + \sigma^{p^e-1}$ is the trace map.

By [65], Lemma 1, there are elements $x_l \in K'$, $1 \le l \le p^e - 1$, such that

$$\nu(x_l) = l \quad \text{and} \quad \nu((\sigma - \mathrm{id}_{K'})x_l) = l + i(l)$$

hold. As $\nu(x_l) = l \ge 0$, they lie actually in R'. Furthermore, together with $1 \in R'$, they give an R-basis for R': To see that they are linearly independent over R, we apply $\sigma - \mathrm{id}_{R'}$ to the linear combination $\lambda_0 \cdot 1 + \sum_{l=1}^{p^e-1} \lambda_l x_l = 0$ with $\lambda_0, \ldots, \lambda_{p^e-1} \in R$ and then take its valuation yielding

$$\infty = \nu(0) = \nu\left(\sum_{l=1}^{p^e-1} (\sigma - \mathrm{id}_{R'})(\lambda_l x_l)\right) = \min_{\lambda_l \neq 0}\{\nu(\lambda_l) + l + i(l)\}.$$

Here, we used the assumption $\nu((\sigma - \mathrm{id}_{K'})x_l) = l + i(l)$ and the fact that $l + i(l) \not\equiv l' + i(l')$ modulo p^e holds (cf. [65], proof of Lemma 1.3) to make the estimation of the valuation of a sum sharp. Thus, the set of l such that $\lambda_l \neq 0$ is empty and the linear independence follows. As R' is a finite R-module of rank p^e by Proposition 5.10, the generating set property follows.

Studying the previous argument, we actually showed that the elements

$$y_l = (\sigma - \mathrm{id}_{R'})x_l, \quad 1 \le l \le p^e - 1$$

are linearly independent in $(\sigma - \mathrm{id}_{R'})R'$ over R and therefore form an R-basis.

Now set $n_l = \left\lfloor \frac{l+i(l)}{p^e} \right\rfloor$ for $1 \le l \le p^e - 1$ and choose a uniformizer $u \in R$. We define

$$z_l = \frac{y_l}{u^{n_l}} = (\sigma - \mathrm{id}_{R'})\frac{x_l}{u^{n_l}}.$$

It is an element of $(\sigma - \mathrm{id}_{K'})K'$, but it also lies in R', as we have

$$\nu(z_l) = \nu(y_l) - n_l \cdot \nu(\pi) = l + i(l) - \left\lfloor p^{-e}(l + i(l)) \right\rfloor \cdot p^e.$$

This is exactly the non-negative rest one obtains when doing division with remainder. Hence, $\nu(z_l) \ge 0$ for $1 \le l \le p^e - 1$, and the z_l are elements of $R' \cap (\sigma - \mathrm{id}_{K'})K'$. In fact, we have $R'[\mathrm{Tr}] = R' \cap (\sigma - \mathrm{id}_{K'})K'$: To see the equality, the inclusion $\supset$ is immediate from $\mathrm{Tr} \circ (\sigma - \mathrm{id}_{K'}) = 0$. For the inclusion $\subset$, we only have to check if $K'[\mathrm{Tr}] \subset (\sigma - \mathrm{id}_{K'})K'$ holds. As $\dim_K K'[\mathrm{Tr}] = \dim_K (\sigma - \mathrm{id}_{K'})K' = p^e - 1$ holds and we have one inclusion, this is true.

We already observed that $\nu(z_l) = l + i(l) \not\equiv l' + i(l') = \nu(z_{l'}) \bmod p^e$ for $l \neq l'$, so the same argument as for the x_l shows that the z_l are an R-basis. We conclude that $u^{n_l} z_l = y_l = 0$ in $R'[\mathrm{Tr}]/(\sigma - \mathrm{id}_{R'})R'$, and this gives us the claimed structure of $H^1(G, R')$. $\square$

Corollary 5.22. *If K'/K is wildly ramified, then*

$$H^1(G, R') \neq \{0\}.$$

Proof. If K'/K is wildly ramified, we consider the summand corresponging to $l = p^e - 1$ in the theorem: We have

$$\left\lfloor \frac{p^e - 1 + i(p^e - 1)}{p^e} \right\rfloor \neq 0 \quad \Longleftrightarrow \quad i(p^e - 1) > 0.$$

But now $i(p^e - 1) \geq 1$. Hence, $H^1(G, R')$ is non-trivial. $\square$

Corollary 5.23. *Let G_1 be cyclic of order p^e and G/G_1 be cyclic of order n. Then*

$$H^1(G, R') \simeq \bigoplus_{l=1}^{p^e-1} R/\mathfrak{m}^{\tilde{n}_l} R,$$

where $\tilde{n}_l = \lfloor (l + i(l) + (n-1)p^e)(np^e)^{-1} \rfloor \leq n_l$ holds. The numbers $i(l)$ and n_l are defined as in Sen's Theorem for the wildly ramified extension corresponding to the extension K'/K'^{G_1}.

Proof. Applying Proposition 5.19, we only have to check if

$$(R'^{G_1}/\widetilde{\mathfrak{m}}^{n_l})^{G/G_1} = R/\mathfrak{m}^{\tilde{n}_l}$$

holds, where $\widetilde{\mathfrak{m}}$ is the maximal ideal in R'^{G_1}. For any positive integer z, we take the G/G_1-invariants of the exact sequence

$$0 \longrightarrow \widetilde{\mathfrak{m}}^z \longrightarrow R'^{G_1} \longrightarrow R'^{G_1}/\widetilde{\mathfrak{m}}^z \longrightarrow 0.$$

Now n is prime to p by Corollary 5.17 and thus invertible in R'^{G_1}. Hence $H^1(G/G_1, \widetilde{\mathfrak{m}}^z)$ is trivial by Proposition 5.9. Therefore, $(R'^{G_1}/\widetilde{\mathfrak{m}}^{n_l})^{G/G_1} = R/\mathfrak{m}^{1+\lfloor \frac{n_l - 1}{n} \rfloor}$ holds by Lemma

5.20. One then computes

$$
\begin{aligned}
1 + \left\lfloor \frac{n_l - 1}{n} \right\rfloor &= \left\lfloor \frac{n_l - 1 + n}{n} \right\rfloor \\
&= \left\lfloor \frac{\left\lfloor \frac{l+i(l)+(n-1)p^e}{p^e} \right\rfloor}{n} \right\rfloor \\
&= \left\lfloor \frac{l + i(l) + (n-1)p^e}{np^e} \right\rfloor .
\end{aligned}
$$

To compare n_l and $\tilde{n}_l$, we obtained

$$
n_l - \tilde{n}_l = n_l - 1 - \left\lfloor \frac{n_l - 1}{n} \right\rfloor ,
$$

and this is easily seen to be non-negative. □

Example 5.24. For a complete discrete valued field K, we consider the Artin–Schreier extension $K' = K[x]/(x^p - x - t)$, where ν_K is a normalized discrete valuation with $\nu_K(t) = -m$ for a positive integer m coprime to p. This is a totally ramified extension of degree p with cyclic Galois group, cf. Example 1.10. We want to compute $H^1(G, R')$, where R and R' are the rings of integers in K and K' with uniformizers u and u'. Let α be a root of $x^p - x - t$ and a, b integers such that $a(-m) + bp = 1$. Denoting ν_L the valuation induced by R', we have $\nu_L(\alpha) = -m$ and $\nu_L(u) = p$. Therefore, $\nu_L(\alpha^a u^b) = 1$ and $\alpha^a u^b$ is a uniformizer in R'. Taking an automorphism $\sigma \in G$ with $\sigma(\alpha) = \alpha + 1$, we compute

$$
\begin{aligned}
\nu_L(\sigma(\alpha^a u^b) - \alpha^a u^b) &= \nu_L(((\alpha + 1)^a - \alpha^a)u^b) \\
&= pb + \nu_L\Big(\sum_{l=1}^{a} \binom{a}{l} \alpha^{a-l} \Big) \\
&= pb + \nu_L(\alpha^{a-1}) \\
&= pb - m(a - 1) \\
&= m + 1.
\end{aligned}
$$

Hence, $\sigma \in G_m \setminus G_{m+1}$. Thus G_{m+1} is trivial. Writing $m = dp + b$ for $d \geq 0$ and $1 \leq b \leq p - 1$, we deduce from Theorem 5.21

$$
H^1(G, R') = (R/\mathfrak{m}^{d+1})^{\oplus b} \oplus (R/\mathfrak{m}^d)^{\oplus p-b-1}.
$$

Chapter 6

Computing the torsion via group cohomology

Given a torsor X_K under an abelian variety A_K, we spend a lot of effort to construct a regular proper model $f\colon X \to S$ of X_K as a certain quotient. This will allow us to apply the Grothendieck spectral sequence for spaces with a group action, gaining information on the torsion appearing in $R^1 f_* \mathscr{O}_X$. In the next sections, we will recapitulate the fundamentals of this spectral sequence following the original source [21], Chapter 5, and apply it to our situation.

6.1 The Grothendieck spectral sequence for schemes with a group action

Let X be a scheme and G be a group acting continuously on X. Assume that the *geometric quotient* $f\colon X \to Y = X/G$ of X under a group G exists, i.e. Y carries the quotient topology, the fibers of f are exactly the G-orbits and $\mathscr{O}_Y \to (f_*\mathscr{O}_X)^G$ is an isomorphism. Then we want to consider *G-sheaves* $\mathscr{F}$ on X and compute the cohomology of $(f_*\mathscr{F})^G$. We use the language of schemes as it fits our situation, but the theory works more general on topological spaces. To make sense of it, we first define the objects (e.g. [8], Definition 5.1):

Definition 6.1. Let $\mathscr{F}$ be an abelian sheaf on X. We call a collection $(\psi_g)_{g\in G}$ of sheaf homomorphisms $\psi_g\colon g_*\mathscr{F} \to \mathscr{F}$ a *G-linearization* if $\psi_e\colon \mathscr{F} \to \mathscr{F}$ is the identity and $\psi_{gh} = \psi_g \circ g_*(\psi_h)$ holds for all $g, h \in G$. That means that the following diagram commutes:

$$\begin{tikzcd} g_*h_*\mathscr{F} \arrow[r, "g_*\psi_h"] \arrow[d, equal] & g_*\mathscr{F} \arrow[r, "\psi_g"] & \mathscr{F} \\ (g\circ h)_*\mathscr{F} \arrow[urr, "\psi_{gh}"'] & & \end{tikzcd}$$

We call $\mathscr{F}$ together with a linearization a *G-sheaf*. We call a homomorphism of abelian sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ a *homomorphism of G-sheaves* if it commutes with the G-linearizations.

Note that in general, there may exist more than one linearization. Furthermore, the structure sheaf $\mathscr{O}_X$ comes equipped with a canonical G-linearization induced by the action of G: Every $g \in G$ acts as an automorphism of ringed spaces, hence gives an isomorphism $\mathscr{O}_X \to g_*\mathscr{O}_X$ with inverse ψ_g. The $(\psi_g)_{g\in G}$ define then a linearization of $\mathscr{O}_X$. In the following discussion, when we consider $\mathscr{O}_X$ as a G-sheaf, we will mean $\mathscr{O}_X$ with its canonical linearization. The following proposition just collects elementary properties of G-sheaves:

Proposition 6.2. *Let X be a scheme and G be a subgroup of automorphisms of X with geometric quotient $Y = X/G$. We denote by $f\colon X \to Y$ the quotient morphism and equip Y with the trivial G-action. Let $\mathscr{F}$ be a G-sheaf on X with G-linearization $(\psi_g)_g$. Then the following statements hold:*

(i) The ψ_g are isomorphisms.

(ii) Each ψ_g induces a bijection of stalks $\mathscr{F}_{g^{-1}(x)} \to \mathscr{F}_x$.

(iii) Each abelian sheaf $\mathscr{G}$ on Y can be pulled back to a G-sheaf $f^*\mathscr{G}$ on X.

Proof. The first statement follows from the equality

$$\psi_g \circ g_*\psi_{g^{-1}} = \psi_{gg^{-1}} = \mathrm{id}_{\mathscr{F}} = \psi_{g^{-1}g} = \psi_{g^{-1}} \circ (g^{-1})_*\psi_g,$$

because the left hand side says that ψ_g is surjective and the right hand side that $(g^{-1})_*\psi_g$ is injective over any open $U \subset X$. Thus, ψ_g is also surjective over any open $U \subset X$ as g^{-1} is an isomorphism on X.

For the second statement, the isomorphism ψ_g induces a bijection $(g_*\mathscr{F})_x \simeq \mathscr{F}_x$ on stalks. Reformulating gives the desired formula.

Now if $\mathscr{G}$ is an abelian sheaf on Y, the pullback $f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ inherits its linearization from the structure sheaf $\mathscr{O}_X$. $\square$

Denote by ($\mathscr{O}_X[G]$-mod) the *category of $\mathscr{O}_X[G]$-modules*, that is to say, $\mathscr{O}_X$-modules $\mathscr{F}$ which are also G-sheaves such that the module structure is compatible with the G-linearizations of $\mathscr{O}_X$ and $\mathscr{F}$ and morphisms being homomorphism of $\mathscr{O}_X$-modules which are also homomorphisms of G-sheaves. Then ($\mathscr{O}_X[G]$-mod) forms an additive category. It behaves even better, see [21], Proposition 5.1.1:

Proposition 6.3. *The category* ($\mathscr{O}_X[G]$-mod) *is an abelian category satisfying $AB5$ and $AB3^*$, admits an generator and thus every $\mathscr{O}_X[G]$-module can be embedded into an injective $\mathscr{O}_X[G]$-module.*

Given a G-sheaf $\mathscr{F}$ on X, the direct image

$$f_*\mathscr{F}\colon (\mathrm{Open}/Y) \to (\mathrm{Set}), \quad V \longmapsto \Gamma(f^{-1}(V), \mathscr{F})$$

is a sheaf on Y. It can naturally be endowed with a G-linearization inherited from $\mathscr{F}$, which is - as G acts trivial on Y - nothing than a group action on $\Gamma(V, f_*\mathscr{F})$ compatible with restrictions. Denote by $\mathscr{F}^G$ the sheaf on Y given by $V \mapsto \Gamma(V, \mathscr{F})^G$. In this sense, we define the following functors:

$$
\begin{aligned}
(\_)^G_Y \colon (G\text{-Sh}/Y) &\longrightarrow (\mathrm{Sh}/Y), \quad \mathscr{F} \longmapsto \mathscr{F}^G \\
f_*^G \colon (G\text{-Sh}/X) &\longrightarrow (\mathrm{Sh}/Y), \quad \mathscr{F} \longmapsto (f_*\mathscr{F})^G.
\end{aligned}
$$

In particular, $f_*^G = (\_)^G_Y \circ f_*$ holds. As Y is a geometric quotient, we have $\mathscr{O}_Y = f_*^G(\mathscr{O}_X)$ and obtain the following commutative diagram

$$
\begin{tikzcd}
(\mathscr{O}_X[G]\text{-mod}) \arrow[r, "f_*"] \arrow[rr, bend left, "f_*^G"] \arrow[d, "\Gamma_X"] \arrow[drr, "\Gamma_X^G"] & (\mathscr{O}_Y[G]\text{-mod}) \arrow[r, "(\_)^G_Y"] \arrow[dr, "\Gamma_Y^G"] & (\mathscr{O}_Y\text{-mod}) \arrow[d, "\Gamma_Y"] \\
(\Gamma(X, \mathscr{O}_X)[G]\text{-mod}) \arrow[rr, "(\_)^G_{\mathrm{Spec}(\mathbb{Z})}"'] & & (\Gamma(Y, \mathscr{O}_Y)\text{-mod}),
\end{tikzcd}
$$

where we consider $(\Gamma(X, \mathscr{O}_X)[G]\text{-mod})$ as the category of G-sheaves over $\mathbb{Z}$ which are $\Gamma(X, \mathscr{O}_X)$-modules compatible with the G-linearization. This lets us write Γ_X^G as at least three different compositions:

$$
\begin{aligned}
\Gamma_X^G &= \Gamma_Y \circ f_*^G, \\
\Gamma_X^G &= (\_)^G \circ \Gamma_X, \\
\Gamma_X^G &= \Gamma_Y^G \circ f_*.
\end{aligned}
$$

These functors behave very well, cf. [21], paragraph above Proposition 5.1.3 and Corollaire:

Proposition 6.4. *Restricting above functors on subcategories (with the same notation), we get:*

(i) $f_* \colon (\mathscr{O}_X[G]\text{-mod}) \to (\mathscr{O}_Y[G]\text{-mod})$ *is a covariant, additive, left exact functor.*

(ii) $f_*^G \colon (\mathscr{O}_X[G]\text{-mod}) \to (\mathscr{O}_Y\text{-mod})$ *is a covariant, additive, left exact functor.*

(iii) $f_* \colon (\mathrm{Ab}/X) \to (\mathrm{Ab}/Y)$ *transforms injective objects into injective objects.*

(iv) The functor $\Gamma_X \colon (\mathrm{Ab}/X) \to (G\text{-mod})$, $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ *sends injective objects to injective objects.*

(v) For all injective abelian sheaves $\mathscr{F}$*, the sheaf* $f_*^G(\mathscr{F})$ *on* Y *is flabby.*

This means that we can apply the Grothendieck spectral sequence. We denote the derived functors of Γ_X^G and f_*^G as follows:

$$H^n(X; G, \mathscr{F}) = R^n\Gamma_X^G(\mathscr{F})\colon (\mathscr{O}_X[G]\text{-mod}) \longrightarrow (\Gamma(Y, \mathscr{O}_Y)\text{-mod}),$$
$$\mathscr{H}^n(G, \mathscr{F}) = R^n f_*^G(\mathscr{F})\colon (\mathscr{O}_X[G]\text{-mod}) \longrightarrow (\mathscr{O}_Y\text{-mod}).$$

This makes us able to formulate [21], Theorem 5.2.2:

Theorem 6.5. *In the situation above, we have three spectral sequences*

$$\mathrm{I}_2^{p,q} = H^p(Y, \mathscr{H}^q(G, \mathscr{F})) \Longrightarrow H^{p+q}(X; G, \mathscr{F}),$$
$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X, \mathscr{F})) \Longrightarrow H^{p+q}(X; G, \mathscr{F}),$$
$$\mathrm{III}_2^{p,q} = H^p(Y; G, R^q f_*\mathscr{F})) \Longrightarrow H^{p+q}(X; G, \mathscr{F}).$$

In the following, we list some important special cases of this theorem where one of the spectral sequence degenerates.

Corollary 6.6. *Let $\mathscr{F}$ be an abelian G-sheaf such that $R^q f_*(\mathscr{F}) = 0$ for all $q > 0$. Then the natural edge homomorphisms from the spectral sequence $\mathrm{III}_2^{p,q}$*

$$H^n(Y; G, f_*(\mathscr{F})) \xrightarrow{\sim} H^n(X; G, \mathscr{F})$$

are isomorphisms; moreover we have

$$\mathscr{H}^n(G, \mathscr{F}) = R^n(\_)_Y^G(f_*\mathscr{F})$$

in $(\mathscr{O}_Y\text{-mod})$.

Corollary 6.7. *Suppose that $\mathscr{H}^q(G, \mathscr{F}) = 0$ for $q > 0$. Then we obtain for $n \geq 0$ canonical identifications $H^n(X; G, \mathscr{F}) = H^n(Y, f_*^G(\mathscr{F}))$ and*

$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X, \mathscr{F})) \Longrightarrow H^{p+q}(Y, f_*^G(\mathscr{F})).$$

In particular, the condition is satisfied for $\mathscr{F} = \mathscr{O}_X$ if X is integral, G finite and f étale.

Proof. The first statement follows again from the degeneration of the spectral sequence $\mathrm{II}_2^{p,q} \Rightarrow H^{p+q}(Y, f_*^G(\mathscr{F}))$. Now prove the additional statement. As G is finite, the quotient morphism is finite and $R^q f_*\mathscr{O}_X = 0$ for $q \geq 1$. By Corollary 6.6, the equality $\mathscr{H}^q(G, \mathscr{F}) = R^q(\_)_Y^G(f_*\mathscr{F})$ holds and hence $\mathscr{H}^q(G, \mathscr{F})_y = H^q(G, (f_*\mathscr{F})_y)$ for every $y \in Y$. We therefore can assume $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(R)$ affine, with R the spectrum of a local ring. If we show that A is a projective $R[G]$-module, we are done. By [47], Proposition II.2 and II.3, this is equivalent to A being a projective R-module and the surjectivity of the trace map $\mathrm{Tr}\colon A \to R$. The projectivity of A as an R-module follows from the flatness of f. To show that the trace is surjective, we have to show that its image is not contained in the maximal ideal $\mathfrak{m}$ of the local ring R. As the trace commutes with reduction modulo $\mathfrak{m}$, it suffices to show that $\mathrm{Tr}\colon A/\mathfrak{m} \to R/\mathfrak{m} = k$ is surjective. Note that because f is unramified, points over the maximal ideal $\mathfrak{m}$ make up an orbit of maximal ideals, all with the same residue field l. Hence, $A/(\mathfrak{m}A)$ is a copy of fields $l^{\oplus r}$. The field extension l over k is separable. Even better, the extension l/k is Galois and G surjects to its Galois group, cf. [66], Chapter I, §7, Proposition 20. So choose an element $\lambda \in l$ with non-trivial trace under $\mathrm{Gal}(l/k)$. Then the trace of $(l, 0, \ldots, 0)$ under G will be the trace of l under $\mathrm{Gal}(l/k)$. Hence, $\mathrm{Tr}\colon A/\mathfrak{m} \to k$ is surjective. $\square$

Corollary 6.8. *Assume that $H^q(X, \mathscr{F}) = 0$ for all $q > 0$. Then*

$$H^n(X; G, \mathscr{F}) = H^n(G, \Gamma(X, \mathscr{F}))$$

and $\mathrm{I}_2^{p,q} = H^p(Y, \mathscr{H}^q(G, \mathscr{F})) \Rightarrow H^{p+q}(G, \Gamma(X, \mathscr{F}))$.

6.2 General computing strategy

We want to apply the previously presented Grothendieck spectral sequences to the situation where we constructed torsors of abelian varieties as quotients, see Proposition 3.22. Let S be the spectrum of a complete discrete valuation ring R and $f\colon X \to S$ be an abelian fibration that is induced by taking the quotients on an abelian fibration $f'\colon X' \to S'$ by a group G, i.e. we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S. \end{array}$$

Proposition 5.10 ensures that the Galois extension $S' = \mathrm{Spec}(R')$ over S is again the spectrum of a complete discrete valuation ring R'. We apply the five-term exact sequences resulting from Theorem 6.5 to two special situations:

Proposition 6.9. *Keep the notation above.*

(i) If we assume h étale and that $H^0(X', \mathscr{O}_{X'})$ and $H^1(X', \mathscr{O}_{X'})$ are free modules over R', then $H^1(X, \mathscr{O}_X) = R^{\oplus g} \oplus H^1(G, R')$ holds, where $g = h^1(\mathscr{O}_{X_\eta})$.

(ii) If we assume $|G|$ invertible in R', then $H^i(X, \mathscr{O}_X) = H^i(X', \mathscr{O}_{X'})^G$ holds for all integers $i \geq 0$.

Proof. We first show (i): To compute the free part of the cohomology groups, we note that cohomology commutes with flat base change, yielding $H^1(X, \mathscr{O}_X) \otimes_R K = H^1(X_\eta, \mathscr{O}_{X_\eta})$. Thus, the free part has dimension $h^1(\mathscr{O}_{X_\eta})$.

For the torsion part of $H^1(X, \mathscr{O}_X)$, we consider the first spectral sequence from Theorem 6.5, namely $\mathrm{I}_2^{r,s} = H^r(X, \mathscr{H}^s(G, \mathscr{O}_{X'})) \Rightarrow H^{r+s}(X'; G, \mathscr{O}_{X'})$. By assumption, h is étale and therefore $\mathscr{H}^s(G, \mathscr{O}_{X'})$ is trivial for $s \geq 1$. Thus, the spectral sequence degenerates and we have $H^r(X, \mathscr{O}_X) = H^r(X'; G, \mathscr{O}_{X'})$ for $r \geq 0$.

Let us take a look at the spectral sequence $\mathrm{II}_2^{r,s} = H^r(G, H^s(X', \mathscr{O}_{X'})) \Rightarrow H^{r+s}(X, \mathscr{O}_X)$: Its five-term exact sequence yields

$$0 \longrightarrow H^1(G, H^0(X', \mathscr{O}_{X'})) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^1(X', \mathscr{O}_{X'})^G.$$

The module $H^1(X', \mathscr{O}_{X'})^G$ is torsion free, so all of the torsion in $H^1(X, \mathscr{O}_X)$ must map to zero. On the other hand, $H^1(G, H^0(X', \mathscr{O}_{X'})) = H^1(G, R')$ is a torsion module, so it must coincide with the torsion part of $H^1(X, \mathscr{O}_X)$.

To compute (ii), we again observe the vanishing of cohomologies: as $|G|$ is invertible in $\mathscr{O}_{X'}$, the cohomology groups $H^r(G, H^s(X', \mathscr{O}_{X'}))$ and $\mathscr{H}^s(G, \mathscr{O}_{X'})$ vanish for $s \geq 1$ due to Proposition 5.9 (the latter can be checked on stalks to obtain usual group cohomology on which to apply the proposition). Thus, the Grothendieck spectral sequences $\mathrm{I}_2^{r,s}$ and $\mathrm{II}_2^{r,s}$ degenerate and we get canonical isomorphisms

$$H^q(X, \mathscr{O}_X) \overset{\mathrm{I}_2^{r,s}}{=\!=} H^q(X'; G, \mathscr{O}_{X'}) \overset{\mathrm{II}_2^{r,s}}{=\!=} H^q(X', \mathscr{O}_{X'})^G.$$

□

Remark 6.10. The situation that $H^i(X', \mathscr{O}_{X'})$ is torsion free for all $i \geq 0$ is satisfied if X' is an abelian scheme: If A_K is an abelian variety, its cohomology groups $H^i(A_K, \mathscr{O}_{A_K})$ are known to have dimension $\binom{g}{i}$ for $i \geq 0$ and $g = \dim A_K$, cf. [56], Corollary 2 on p. 129. Now if $A_{K'}$ has good reduction, its special fiber $A_{s'}$ is also an abelian variety and the morphisms φ^i from Theorem 2.9 are isomorphisms, i.e. no torsion appears.

Another important case where this is satisfied is the case of elliptic fibrations with sections: The section forces the multiplicity of the closed fiber to be one and therefore has no torsion according to Proposition 2.7.

At first sight, this proposition only helps in describing the torsion part in the cohomology of quotients. But the following theorem by Chatzistamatiou and Rülling (cf. [10], Theorem 1) helps in comparing the cohomology groups of models of the generic fiber:

Theorem 6.11. *Let $f\colon X \to Y$ be a projective and birational morphism between excellent and regular schemes. Then the higher direct images of $\mathscr{O}_X$ under f vanish, i.e. $R^i f_* \mathscr{O}_X = 0$ for all $i > 1$.*

This theorem implies that the spectral sequence $H^r(Y, R^s f_* \mathscr{O}_X) \Rightarrow H^{r+s}(X, \mathscr{O}_X)$ degenerates, i.e. $H^r(Y, f_*\mathscr{O}_X) = H^r(X, \mathscr{O}_X)$ holds. If both X and Y are additionally integral, then f is connected and we have $f_*\mathscr{O}_X = \mathscr{O}_Y$ by a version of Zariski's Main Theorem (cf. [26], Corollary III.11.4). In particular, this applies to a projective morphism $f\colon X \to \tilde{X}$ of two regular models of X_K over S. Even if S is not excellent, we can make the base change to the completion of the stalk of a closed point $s \in S$. As we have seen in Proposition 2.1, it does not change the torsion structure. We therefore can assume that X and Y are excellent and apply Theorem 6.11.

In general, one cannot hope for the existence of a morphism between two regular proper models, but in the case of fibered surfaces, there always exists a regular fibered surface dominating any two other regular ones (cf. [45], Lemma 9.3.20). The morphisms are proper, but if we restrict the surfaces to an open affine base, Lichtenbaum's Theorem (cf. [42], Chapter I, Theorem 2.8) grants that they are indeed projective. Again, we apply Theorem 6.11 to show that the cohomology groups of regular fibered surfaces are isomorphic. For general abelian fibrations, the existence of a regular model dominating two given regular ones is unknown. For reference, we state the conclusion:

Proposition 6.12. *Let A_K be an abelian variety over K with torsor X_K. Assume that for any two projective regular models X_1 and X_2 of X_K over S, there is a projective regular model X_0 of X_K over S that dominates both X_1 and X_2, i.e. there is a commutative diagram*

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & S & \end{tikzcd}$$

of X_K-models. Then $H^i(X_1, \mathscr{O}_{X_1}) = H^i(X_0, \mathscr{O}_{X_0}) = H^i(X_2, \mathscr{O}_{X_2})$ holds for $i \geq 0$. The conditions of this proposition are satisfied if S is affine and A_K an elliptic curve.

Remark 6.13. In the situation of Proposition 6.9 (i), if X_K is of dimension greater than one, we can ask for the structure of the higher direct images. We note that in fact, there is a seven term exact sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker(H^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}.$$

If $E_2^{0,2} = H^2(X', \mathscr{O}_{X'})$ is a free R'-module, then the torsion $\mathscr{T}_2$ in $H^2 = H^2(X, \mathscr{O}_X)$ is mapped to zero. As $E_2^{2,0} = H^2(G, H^0(X', \mathscr{O}_{X'}))$ and $E_2^{1,1} = H^1(G, H^1(X', \mathscr{O}_{X'}))$ are also torsion modules, $\ker(H^2 \to E_2^{0,2})$ must be exactly the torsion part $\mathscr{T}_2$, and it sits inside the exact sequence

$$H^2(G, H^0(X', \mathscr{O}_{X'})) \longrightarrow \mathscr{T}_2 \longrightarrow H^1(G, H^1(X', \mathscr{O}_{X'})).$$

It would be interesting if one could get better information from the spectral sequence.

Chapter 7

The torsion in the cohomology of abelian fibrations as quotients

This chapter puts the theory we have developed together to compute the torsion in the cohomology of abelian fibrations, and therefore its theorems are at the heart of this thesis. As the torsion in the cohomology does not change under flat base change, it suffices to study the case where S is the spectrum of a complete discrete valuation ring R. This will be our general assumption. Moreover, we assume that the residue field k of R is algebraically closed. We denote by p the characteristic of k and by K the field of fraction of R. As usual, A_K will be an abelian variety over K and X_K an torsor under A_K.

7.1 The case of abelian varieties with good reduction and elliptic curves with multiplicative reduction

Let A_K have good reduction. In Section 3.3, we constructed a proper regular model X of X_K over S as follows: Take a minimal Galois extension of K'/K so that $X_K(K')$ is non-empty and let S' be the normalization of S in K'. Then $\mathrm{Gal}(K'/K)$ acts on the Néron model A' of $A_{K'}$ over S', and the structure morphism is equivariant with respect to this action. Taking the quotients of A' and S', we obtained a proper regular model $X \to S$ with structure morphism induced by $A' \to S'$. To study the torsion in $R^1 f_* \mathscr{O}_X$, we want to apply Proposition 6.9. Yet, the quotient morphism $A' \to X$ is not étale in general. To see when the quotient morphism is étale or how to circumvent occurring problems, we need to get a better understanding of the group $H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), A(K^{\mathrm{sep}}))$. In the case of elliptic curves, we will be able to compute the torsion structure if the special fiber A_k is an ordinary elliptic curve or A_K has multiplicative reduction, cf. Theorem 7.3 and 7.5. We are going to reproduce some of the arguments of [62], Théorème 9.4.1, clarifying the group structure of $H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), A(K^{\mathrm{sep}}))$. It uses the theory of p-divisible groups, e.g. [74].

For the rest of the discussion, we denote $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and assume that the order d of $[X_K]$ in $H^1(G, A(K^{\mathrm{sep}}))$ is a power of p, i.e. $d = p^e$. This can easily be achieved as follows: Using the same notation as in Section 4.2, we define the index of X_K to be $m = \delta(X_K/K)$, the greatest common divisor of all field extension degrees of closed points in X_K. By [41], Proposition 5, d divides m, and they have the same prime factors. Given a proper regular model of X_K over S, every field extension K'/K of degree prime to p is Galois (Corollary 5.13) and induces an étale covering $\pi_X \colon X' \to X$, see Corollary 4.6. Therefore, using Proposition 6.9, we obtain $H^i(X, \mathscr{O}_X) = H^i(X', \mathscr{O}_{X'})^{G'}$ for $i \geq 0$, where $G' = \mathrm{Gal}(K'/K)$. It thus suffices to study the p-part of a splitting field.

In Section 3.3, we already considered the reduction map $A(K) \to A(k)$ (note that every closed point of A lies in A_k, so that $A(k) = A_k(k)$ holds), which sits inside a short exact sequence

$$0 \longrightarrow A_1(S) \longrightarrow A(K) \longrightarrow A(k) \longrightarrow 0.$$

Here, for the moment, $A_1(S)$ simply denotes the kernel of the reduction map, but we will soon consider A_1 as a p-divisible group. Note that for elliptic curves, $A_1(S)$ is part of a filtration $\ldots \subset A_2(S) \subset A_1(S) \subset A(S)$, cf. [73], Theorem 4.2. Applying the Néron mapping property to any algebraic extension K' over K gives a similar exact sequence. Moreover, if K'/K is Galois, we obtain an exact sequence of $\mathrm{Gal}(K'/K)$-modules, where $\mathrm{Gal}(K'/K)$ acts trivially on $A(k)$. In particular, denoting by S^{sep} the normalization of S in K^{sep}, we get a short exact sequence

$$0 \longrightarrow A_1(S^{\mathrm{sep}}) \longrightarrow A(K^{\mathrm{sep}}) \longrightarrow A(k) \longrightarrow 0$$

of G-modules, from which we can take the G-invariants. If we equip $A(K^{\mathrm{sep}})$ with the discrete topology, it becomes a discrete G-module as in Example 5.2. The same is true for $A(k)$ because G acts trivially upon it, making the continuity of the group action easy to see. Imposing on $A_1(S^{\mathrm{sep}})$ the discrete topology as well, this yields a long exact sequence of discrete G-modules. We are interested in its first cohomology groups. As $A(K^{\mathrm{sep}}) \to A(k)$ is surjective, this yields the exact sequence

$$0 \longrightarrow H^1(G, A_1(S^{\mathrm{sep}})) \longrightarrow H^1(G, A(K^{\mathrm{sep}})) \longrightarrow H^1(G, A(k)) \longrightarrow H^2(G, A_1(S^{\mathrm{sep}})).$$

The term $H^1(G, A(k))$ is easy to understand: It is equal to the group of *continuous* group homomorphisms $\mathrm{Hom}(G, A(k))$ by the cocycle description of the first cohomology group, see Lemma 5.3. The groups appearing in the sequence are torsion groups (Proposition 5.1), so taking the p-torsion part is exact. Writing $\mathrm{Hom}(G, A(k))[p^\infty]$ for the p-primary part of $\mathrm{Hom}(G, A(k))$, we have an equality $\mathrm{Hom}(G, A(k))[p^\infty] = \mathrm{Hom}(G, A(k)[p^\infty])$ as follows:

Any $\phi \colon G \to A(k)$ annihilated by p^n implies $p^n \phi(\sigma) = 0$, hence every image element $\phi(\sigma)$ is of finite order and ϕ factors over $A(k)[p^\infty]$. Conversely, note that the kernel of ϕ is open in G due to $A(k)$ discrete and ϕ continuous. Therefore, it factors over the finite quotient $G/\ker(\phi)$, and the image of ϕ is a finite subgroup of $A(k)[p^\infty]$. Hence, ϕ is annihilated by a power of p.

We now use that A_K has good reduction and that k is algebraically closed, so A_k is again an abelian variety of dimension g with $A(k)[p^\infty] = \varinjlim_n (\mathbb{Z}/p^n\mathbb{Z})^{\oplus f} = (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus f}$, where f is called *the p-rank* or *f-rank* of A_k, cf. [56], p.147. It suffices the inequality $0 \leq f \leq g$. In particular, we obtain

$$\mathrm{Hom}(G, A(k)) = \mathrm{Hom}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus f}).$$

Now to understand the kernel of the reduction map is more difficult. In fact, A_1 is a *p-divisible group*, cf. [27], Theorem C.2.6 and [68], Proposition on p. 61. That means the following:

Definition 7.1. Let $\Gamma = (\Gamma_n, i_n)$ be a direct system of finite group schemes Γ_n over R, where n runs through the non-negative integers. Moreover, let h be a fixed non-negative integer. We call Γ a *p-divisible group over R of height h* if it satisfies the following two properties:

(i) Γ_n is a a finite group scheme of order p^{nh}, i.e. it is locally free of rank p^{nh} over R.

(ii) For each $n \geq 0$, the sequence

$$0 \longrightarrow \Gamma_n \xrightarrow{i_n} \Gamma_{n+1} \xrightarrow{p^n} \Gamma_{n+1}$$

is exact, i.e. Γ_n can be identified via i_n with the kernel of multiplication by p^n on Γ_{n+1}.

For example, given the multiplicative group $\mathbb{G}_m$ over R, one takes the kernel μ_{p^n} of the multiplication $p^n\colon \mathbb{G}_m \to \mathbb{G}_m$. This yields a p-divisible group $\mathbb{G}_m(p) = (\mu_{p^n}, i_n)$ of height 1, where $i_n\colon \mu_{p^n} \to \mu_{p^{n+1}}$ is the inclusion. The same construction applies to the abelian scheme A over R in our case. Denoting $A[p^n]$ the kernel of the multiplication by p^n, we obtain the p-divisible group $(A[p^n], i_n)$ of height $2g$, where g is the dimension of A_K. In case that A_k is an ordinary abelian variety, i.e. the p-rank of A_k equals its dimension g, the finite group scheme $A[p^n]$ has connected component isomorphic to $\mu_{p^n}^{\oplus g}$, and A_1 can be identified with $\mathbb{G}_m(p)^{\oplus g}$. For a complete algebraic extension K'/K with normalization R' of R in K', and $S' = \mathrm{Spec}(R')$, one defines for a p-divisible group $\Gamma = (\Gamma_n, i_n)$ the S'-valued points of Γ by

$$\Gamma(S') = \Gamma(R') = \varprojlim_n \Gamma(R'/\mathfrak{m}'^n) = \varprojlim_n \varinjlim_j \Gamma_j(R'/\mathfrak{m}'^n).$$

Note that $\Gamma(S)_{\mathrm{tors}} = \varinjlim_n \Gamma_n(R')$ holds, cf. [74], p. 167. We have $\mathbb{G}_m(p)(R^{\mathrm{sep}}) = 1 + \mathfrak{m}^{\mathrm{sep}}$, which sits inside the exact sequence

$$1 \longrightarrow \mathbb{G}_m(p)(S^{\text{sep}}) \longrightarrow \mathbb{G}_m(S^{\text{sep}}) \xrightarrow{r} \mathbb{G}_m(k) \longrightarrow 1, \tag{7.1}$$

where r is the reduction map. Furthermore, denoting by ν the valuation on K^{sep} that extends uniquely the one on K, we obtain the exact sequence

$$1 \longrightarrow \mathbb{G}_m(S^{\text{sep}}) \longrightarrow \mathbb{G}_m(K^{\text{sep}}) \xrightarrow{r} \mathbb{Q} \longrightarrow 1. \tag{7.2}$$

Both exacts sequences together will show $H^2(G, \mathbb{G}_m(p)(S^{\text{sep}}))[p^\infty] = 0$ as follows: Taking the G-invariants of the exact sequence (7.2) and using $H^1(G, \mathbb{G}_m(K^{\text{sep}})) = 0$ by Hilbert's Theorem 90, we obtain $H^1(G, \mathbb{G}_m(S^{\text{sep}})) = \text{coker}(\nu) = \mathbb{Q}/\mathbb{Z}$. As S^{sep} is strictly Henselian, its Brauer group vanishes (cf. [50], §4, Corollary 1.7).Hence, $H^2(G, \mathbb{G}_m(S^{\text{sep}})) = \text{Br}(S^{\text{sep}})$ is trivial. Therefore, we deduce from sequence (7.1) and

$$H^1(G, \mathbb{G}_m(k))[p^\infty] = \text{Hom}(G, k^\times[p^\infty]) = \text{Hom}(G, \{1\}) = 0$$

that $H^2(G, \mathbb{G}_m(p)(S^{\text{sep}}))[p^\infty] = 0$. As sums of G-modules commute with group cohomology (cf. Proposition 5.1), we obtain the exactness of

$$0 \longrightarrow H^1(G, A_1(S^{\text{sep}}))[p^\infty] \longrightarrow H^1(G, A(K^{\text{sep}}))[p^\infty] \longrightarrow H^1(G, A(k))[p^\infty] \longrightarrow 0.$$

As noted above, $H^1(G, A(k))[p^\infty] = \text{Hom}(G, \mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g}$ holds. Using the surjectivity of the reduction map on S-valued points, exact sequence (7.1) provides us with the equality $H^1(G, \mathbb{G}_m(p)(S^{\text{sep}}))[p^\infty] = H^1(G, \mathbb{G}_m(S^{\text{sep}}))[p^\infty] = \mathbb{Q}_p/\mathbb{Z}_p$ and we obtain

$$H^1(G, A_1(S^{\text{sep}}))[p^\infty] = H^1(G, \mathbb{G}_m(p)(S^{\text{sep}})^{\oplus g})[p^\infty] = (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g}.$$

Given a class $[X_K]$ of a torsor in $H^1(G, A(K^{\text{sep}}))$, the splitting field of its image in $H^1(G, A(k))$ gives us information on the maximal field extension inducing an étale covering:

Lemma 7.2. *Let A_k be an ordinary abelian variety and X be the regular projective model of X_K over S constructed in Proposition 3.22. We denote by ψ the surjection $H^1(G, A(K^{\text{sep}})) \to H^1(G, A(k))$. Then the following holds:*

(i) $\psi([X_K])$ has a unique minimal splitting field. It is Galois, and its Galois group is the sum of at most g cyclic groups.

(ii) The unique minimal splitting field of $\psi([X_K])$ coincides with M, the maximal field extension inducing an étale covering of X (cf. Section 4.2).

Proof. Under the isomorphism $H^1(G, A(k))[p^\infty] = \text{Hom}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g})$, we can identify

$\psi([X_K])$ with a continuous group homomorphism $\varphi\colon G \to (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g}$. Denote its kernel by H. It is an open subgroup and therefore has finite index. By Galois theory, H is given by $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ for some unique finite Galois extension K'/K. As the restriction is given by

$$\mathrm{Hom}(G, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g}) \longrightarrow \mathrm{Hom}(H, (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g}), \quad \phi \longmapsto \phi|_H,$$

the torsor X_K becomes trivialized by a field extension L if and only if $K' \subset L$. Hence uniqueness. The Galois group of the extension K'/K is the sum of at most g cyclic groups because $\mathrm{Gal}(K'/K) = G/H = \mathrm{im}(\varphi) \subset (\mathbb{Q}/\mathbb{Z})^{\oplus g}$.

For (ii), interpret $[X_K]$ as a cocycle ξ with $\xi_\sigma \in A(K^{\mathrm{sep}})$. Then $\psi([X_K])$ corresponds to the cocycle $\overline{\xi}$ given by $\overline{\xi}_\sigma \in A(k)$. Using the same notation for $[X_M]$, we see by Proposition 3.22 that $\psi([X_M]) = 0$, i.e. the splitting field of $\psi([X_K])$ sits inside M. On the other hand, if $\psi([X_K])$ splits over K'/K, then $\psi([X_{K'}]) = 0$, which means $M \subset K'$. $\square$

In the case of elliptic curves where A_k is ordinary, Raynaud proved in [62], Théorème 9.4.1, that X_K is in the image of $H^1(G, A_1(S^{\mathrm{sep}}))[p^\infty] \to H^1(G, A(K^{\mathrm{sep}}))[p^\infty]$ if and only if the relatively minimal elliptic fibration $f\colon X \to S$ associated to X_K is cohomologically flat. By Theorem 2.9, this is equivalent to say that there is no torsion in $R^1 f_* \mathscr{O}_X$. Hence, if A_k is ordinary, the torsion can be computed for every A_K-torsor by Proposition 6.9:

Theorem 7.3. *Let A_K be a g-dimensional abelian variety with good reduction such that A_k is ordinary and let X_K be an A_K-torsor. Denote by M/K the maximal field extension inducing an étale covering of X with Galois group H and by K'/K a minimal Galois extension splitting X_K. Then the following holds:*

(i) If $M = K'$, the multiplicity m of the closed fiber of X equals $[M\colon K]$ and we have $H^1(X, \mathscr{O}_X) = R^{\oplus g} \oplus H^1(H, R')$.

(ii) If additionally A_K is an elliptic curve, write $H' = \mathrm{Gal}(K'/M)$. Then the torsion in $R^1 f_* \mathscr{O}_X$ is isomorphic to

$$H^1(H, R'^{H'}) = \bigoplus_{l=1}^{p^e-1} R/\mathfrak{m}^{\tilde{n}_l} R$$

where $\tilde{n}_l = \lfloor (l + i(l) + (n-1)p^l)(np^l)^{-1} \rfloor$. The positive integer $i(l)$ is defined as in Sen's Theorem for the wildly ramified extension corresponding to the extension K'/K'^{H_1} and $[M\colon K] = np^e$ with n not divisible by p.

Note that in the missing case in which A_K has good reduction and the special fiber A_k is a supersingular elliptic curve, there is torsion in $H^1(X, \mathscr{O}_X)$ according to [62], Théorème 9.4.1, but we are not able to compute it.

Proof. Assume $M = K'$. Then the equality $m = [M\colon K]$ holds by Proposition 4.2. Furthermore, X is the étale quotient $q\colon A' \to X$ constructed in Proposition 3.22, see Corollary 4.5. As the cohomology groups of the abelian scheme A' are torsion-free, we can apply Proposition 6.9 to obtain

$$H^1(X, \mathscr{O}_X) = R^{\oplus g} \oplus H^1(H, R').$$

Now if A_K is an elliptic curve, we first take the quotient $Y = A'/H'$. This is cohomological flat over $S'^{H'}$ according to [62], Théorème 9.4.1. This means that $H^i(Y, \mathscr{O}_Y) = R'^{H'}$ is free and we can apply Proposition 6.9 to the quotient morphism $Y/H \to X$ (this is the map q_2 in the factorization of Proposition 3.22). Hence, we have

$$H^1(X, \mathscr{O}_X) = R \oplus H^1(H, R'^{H'}).$$

In Lemma 7.2, we noticed that H is cyclic. It decomposes into its higher ramification groups $H = H/H_1 \times H_1$ with H/H_1 and H_1 cyclic. Using Proposition 5.19, we obtain

$$H^1(H, R'^{H'}) = H^1(H_1, R'^{H'})^{H/H_1}$$

on which we apply Sen's Theorem 5.23. □

Remark 7.4. Let X_K be a torsor under A_K whose corresponding cocycle class ξ is in the image of $H^1(G, A_1(S^{\mathrm{sep}}))[p^\infty]$ in $H^1(G, A(K^{\mathrm{sep}}))[p^\infty]$. That is, the reduction $r(\xi_\sigma)$ is trivial in $A'(k)$ for all $\sigma \in G$. In the case that A_K is an elliptic curve and A_k an ordinary elliptic curve, we used in the theorem that the regular model $Y = A'/H'$ has no torsion in its cohomology group. It would be interesting to see this fact directly from our construction in Proposition 3.22 and generalize it to higher dimensions. In the same way, we would like to know what happens in the case that A_k is not ordinary, as in that case the reduction also vanishes.

If E_K is an elliptic curve with multiplicative reduction, Liu, Lorenzini and Raynaud showed in [46], §8, that a torsor X_K of E_K has a unique splitting field and that the relatively minimal proper regular model $f\colon X \to S$ is an étale quotient as constructed in Section 3.1. As before, we prove:

Theorem 7.5. *Let X_K be a torsor under an elliptic curve E_K with multiplicative reduction. Then X_K has a unique splitting field K'/K that is a cyclic Galois extension with Galois group H and the torsion in the relatively minimal regular proper model $f\colon X \to S$ is given by*

$$H^1(H, R') = \bigoplus_{l=1}^{p^e-1} R/\mathfrak{m}^{\tilde{n}_l} R,$$

where $\tilde{n}_l = \lfloor (l + i(l) + (n-1)p^l)(np^l)^{-1} \rfloor$*. The positive integer* $i(l)$ *is defined as in Sen's Theorem for the wildly ramified extension corresponding to the extension* K'/K'^{H_1} *and* $[K' \colon K] = np^e$ *with* n *not divisible by* p*.*

7.2 The case of elliptic curves with additive reduction

We use the same notation as in the previous section, except that we assume that A_K does not have semi-abelian reduction, cf. Section 3.2. We would like to reduce the study of the torsion in the cohomology of a proper regular model to the case where A_K has good reduction (or more generally to semi-abelian reduction if we would have solved that case in higher dimension). By Theorem 3.17, there is a unique minimal finite extension L/K such that A_L has semi-abelian reduction. This is a Galois extension and its degree is coprime to each prime greater than $2g+1$. If $p = \mathrm{char}(k)$ is coprime to $d = [L \colon K]$, we call A_K *tamely ramified.* We would like to take the base change to L, take a regular proper model of X_L, compute the torsion part in the first cohomology group of this model and then relate it to X, using Proposition 6.9. But there are obstacles in this consideration. The first two are that we do not know how to relate the torsion in the cohomology of proper regular models and how they behave on base change.

Yet, in the case of elliptic fibrations, this works: Let $f \colon X \to S$ be an elliptic fibration. Then the cohomology group $H^1(X, \mathscr{O}_X)$ is independent of the chosen proper regular model (cf. Proposition 6.12). Moreover, we can resolve singularities. This makes us able to consider the following commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dd, bend right=60, "f'"'] & X'/G \arrow[d, "h"] \arrow[dd, bend left=60] \\ X \times_S S' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] & S, \end{tikzcd} \tag{7.3}$$

where

(i) $S' \to S$ is the unique extension induced by L/K, with Galois group G (cf. Proposition 5.13).

(ii) The lower square is the base change diagram of X along S'.

(iii) $X' \to X \times_S S'$ is obtained by first taking the normalization of $X \times_S S'$ and then taking the *minimal* desingularization, i.e. every other desingularization factors uniquely through X'.

(iv) G acts on $X \times_S S'$ by base change, hence on the normalization and on the minimal desingularization X' via their universal properties. The structure morphism f' is G-equivariant and G acts trivially on X so that the universal property of the quotient gives a map $X'/G \to X$, which is birational in our case.

Proposition 6.9 tells us

$$H^1(X'/G, \mathscr{O}_{X'/G}) = H^1(X', \mathscr{O}_{X'})^G.$$

Furthermore, the morphism h is proper birational and we know that X'/G is a normal scheme. This is too weak to apply Theorem 6.11, but the statement holds for fibrations of surfaces under weaker assumptions, cf. [44], Proposition 1.2:

Proposition 7.6. *Let $f\colon X \to S$ and $g\colon Y \to S$ be fibrations, X normal and Y regular and $h\colon X \to Y$ be a proper, birational morphism of fibrations. Then*

$$H^1(X, \mathscr{O}_X) = H^1(Y, \mathscr{O}_Y)$$

holds.

Proof. By [44], Proposition 1.2, we see that $R^1 h_* \mathscr{O}_X$ vanishes. The isomorphism follows from the Grothendieck spectral sequence applied to $f = g \circ h$. □

Hence, we obtain $H^1(X, \mathscr{O}_X) = H^1(X', \mathscr{O}_{X'})^G$. We now turn to the question when this minimal tame extension is achieved. Katsura and Ueno gave a construction for the global setting where S is a projective Dedekind scheme over an algebraically closed field and X is a relatively minimal elliptic curve over S, cf. [31], Theorem 2.3. The proof is given by explicitly taking equations, making base change, normalizing and blowing up. It holds verbatim in the case that the base scheme is local:

Theorem 7.7. *Let $f\colon X \to S$ be a minimal elliptic fibration with additive reduction. Define u as follows:*

fiber type	I_b^*	II	II^*	III	III^*	IV	IV^*
u	2	6	6	4	4	3	3

We assume that p does not divide u. Then $f'\colon X' \to S'$ from diagram (7.3) *has the wild fiber $X'_{s'} = p^e \mathrm{I}_{2b}$ if $F = \mathrm{I}_b^*$ and $X'_{s'} = p^e \mathrm{I}_0$ else. In particular, the fiber type of a regular minimal model of E_L is of type I_{2b} or I_0 by Theorem 1.13 and E_L has semi-abelian reduction.*

What Katsura and Ueno have done explicitly is implicitly the mechanism of Tate's famous algorithm, only that it works for elliptic fibrations with sections. Tim and Vladimir

Dokchitser have studied the algorithm in [13] to characterize the Kodaira type from the minimality of $\nu(a_i)i^{-1}$, where a_i are the coefficients of a minimal Weierstraß equation. They use this formulation to study the behaviour of the Kodaira type under *tame* extensions, cf. [13], Theorem 3:

Theorem 7.8. *Let K'/K be a tame extension of ramification degree l. Assume that the residue field k is perfect. Then:*

(i) If E_K has Kodaira type I_n, then $E_{K'}$ has Kodaira type I_{ln}.

(ii) If E_K has Kodaira type I_n^*, then $E_{K'}$ has Kodaira type I_{ln}^* if l is odd and type I_{ln} if l is even.

(iii) In all other cases, the type of $E_{K'}$ is determined by

$$\delta_{E_{K'}} \equiv l\delta_{E_K} \mod 12,$$

where $\delta = 0, 2, 3, 4, 6, 8, 9, 10$ if E has Kodaira type $\mathrm{I}_0, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_0^*, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$ respectively.

If k is algebraically closed, each torsor under E_K has the same Kodaira symbol, see [46], Theorem 6.6. Hence, we can read off the theorem that the numbers u defined by Katsura and Ueno are the minimal ones with respect to possible tame extensions, and that in the cases left for $p = 2$ or $p = 3$, there exists no tame extension such that the Kodaira type becomes of good or multiplicative type. We gather the information obtained in the following theorem:

Theorem 7.9. *Let E_K be an elliptic curve with additive reduction such that p does not divide u defined in Theorem 7.7. Let X_K be a torsor under E_K with regular proper model X. Then*

$$H^1(X, \mathscr{O}_X) = H^1(X', \mathscr{O}_{X'})^H,$$

where H is the Galois group of the unique extension L/K of degree u (cf. Corollary 5.13) and X' is a regular proper model of X_L.

Chapter 8

Application to the Kodaira classification

For the main part of this thesis, we have worked with an abelian fibration $X \to S$ such that S is the spectrum of a complete discrete valuation ring. As we saw in Proposition 2.1, we can always restrict ourselves to this case to study the torsion in the cohomology. Whether one can go the other direction is discussed in the first section.

In the second section, we show how elliptic fibrations fit into the Kodaira classification. In particular, if the fibration has a wild fiber, we show which fiber types and multiplicities are possible and can be realized in Kodaira dimension $-\infty$. For Kodaira dimension zero, this was already nearly settled by Bombieri and Mumford [7] and Mitsui [51]. We add the description of the torsion structure and show that two of the three cases for which no example could be given are in fact impossible.

8.1 From local fibrations to global fibrations

Let K be the function field of a Dedekind scheme S proper over an algebraically closed field and A_K be an abelian variety. For every closed point $s \in S$, denote by K_s the field of fractions of $\widehat{\mathscr{O}}_{S,s}$ and by G resp. G_s the Galois groups of K^{sep}/K and K_s^{sep}/K_s. Doing base change by K_s, we obtain a map

$$\psi\colon H^1(G, A_K(K^{\mathrm{sep}})) \longrightarrow \prod_{s \neq \eta} H^1(G_s, A_{K_s}(K_s^{\mathrm{sep}})), \quad X_K \longmapsto X_{K_s}.$$

If A_K is elliptic, ψ factors over the direct sum: By Theorem 3.9, a relatively minimal proper regular model X of X_K over S exists and is unique. If the fiber X_s over s has a rational point, $X \times_S \mathrm{Spec}(\widehat{\mathscr{O}}_{S,s})$ also has a rational point which lifts to a rational point on X_{K_s} by Hensel's Lemma. Now $X \to S$ has only finitely many non-smooth fibers (cf. [19], Théorème 12.2.4), hence only finitely many fibers without rational points. Therefore, all but finitely many torsor X_{K_s} are trivial.

Moreover, $X \times_S \mathrm{Spec}(\widehat{\mathscr{O}}_{S,s})$ is again a relatively minimal elliptic fibration and the Kodaira symbol, multiplicity and torsion structure of X_s and $X \times_S \mathrm{Spec}(\widehat{\mathscr{O}}_{S,s})$ remain the same by Proposition 2.1. So if ψ would surject onto the direct sum, we could construct a relatively minimal elliptic fibration with prescribed Kodaira symbol, multiplicity and torsion at fibers over given points of S. This is indeed often the case, and we are going to sketch why. For a reference, we refer to [12], §5.4 or [49], Chapter III, Theorem 11.6. There is also a generalization to abelian schemes over regular, noetherian, integral, separated base schemes by Keller [33].

Let K be the function field of a Dedekind scheme S over an algebraically closed ground field k and A_K be an abelian variety over K, with Néron model N over S. Denote by $i\colon \operatorname{Spec}(K) \to S$ the inclusion of the generic point. As N is a commutative group scheme, we can consider it as an abelian sheaf on the étale site of S. We apply the Leray–Serre spectral sequence $H^r_{\text{ét}}(S, R^s i_* i^* N) \Rightarrow H^{r+s}_{\text{ét}}(K, i^* N)$ and obtain

$$0 \to H^1_{\text{ét}}(S, i_* i^* N) \to H^1_{\text{ét}}(K, i^* N) \to H^0_{\text{ét}}(S, R^1 i_* i^* N) \to H^2_{\text{ét}}(S, i_* i^* N).$$

Note that $i^* N = A_K$ and $i_* i^* N = N$ hold due to the Néron mapping property and that $H^1_{\text{ét}}(K, A_K)$ is the same as the Galois cohomology group $H^1(G, A_K(K^{\text{sep}}))$, where $G = \operatorname{Gal}(K^{\text{sep}}/K)$. Furthermore, the restriction

$$H^0_{\text{ét}}(S, R^1 i_* i^* N) \longrightarrow \prod_{s \neq \eta} (R^1 i_* i^* N)_{\overline{s}}$$

to the product of stalks at geometric points is injective, cf. [50], Proposition II.2.10. Moreover, denoting by K_s^h the field of fractions of the henselization $\mathscr{O}^h_{S,s}$ of the stalk $\mathscr{O}_{S,s}$ for a closed point $s \in S$ and by $i_s^h\colon \operatorname{Spec}(K_s^h) \to S$ the natural inclusion, we identify

$$(R^1 i_* i^* N)_{\overline{s}} = H^1(K_s^h, (i_s^h)^* N).$$

In fact, using the notation as at the beginning, $H^1(K_s^h, (i_s^h)^* N)$ coincides with the group $H^1(K_s, (i_s)^* N)$ (cf. [12], Remark 5.4.2), which in turn can be identified with the group $H^1(G_s, A_{K_s}(K_s^{\text{sep}}))$. Arguing further, one puts this together to obtain the exact sequence

$$0 \to H^1_{\text{ét}}(S, N) \to H^1(G, A_K(K^{\text{sep}})) \to \bigoplus_{s \neq \eta} H^1(G_s, A_{K_s}(K_s)) \to H^2_{\text{ét}}(S, N).$$

Note that the group $H^1_{\text{ét}}(S, N)$ is also known as the *Tate–Shafarevich group of* N. In the case that $A_K = E_K$ is an elliptic curve, we get the following result (cf. [12], Corollary 5.4.6):

Proposition 8.1. *Let E_K be an elliptic curve over K. Take the relatively minimal proper regular S-model $f\colon E \to S$ of E_K and assume that the elliptic fibration f is nontrivial, i.e. $E \not\cong E_k \times_k S$. Then the global-to-local map $H^1(G, E_K(K^{\text{sep}})) \to \bigoplus H^1(G_s, E_{K_s}(K_s^{\text{sep}}))$, where the sum runs over all closed points $s \in S$, is surjective.*

8.2 Global invariants

For the whole section, let S be a proper Dedekind scheme over an algebraically closed field k of characteristic p and $f\colon X \to S$ be a relatively minimal genus-1-fibration. By Zariski–Goodman (e.g. [4], Theorem 1.28, or [42], Chapter I, Theorem 2.8), X/k is a projective surface. There is a very famous formula

$$\chi(\mathscr{O}_X) = \frac{1}{12}(c_2(X) + (K_X \cdot K_X)),$$

called the *Noether Formula* (cf. [4], Chapter 5, p.69), where $c_2(X)$ denotes the second Chern class of X. It is equal to the l-adic Euler characteristic $e(X) = \sum_{i=0}^{4}(-1)^i b_i(X)$, where $b_i(X) = \dim_{\mathbb{Q}_l} H^i(X_{\text{ét}}, \mathbb{Q}_l)$ is the i-th l-adic Betti number (l prime to p). To obtain more information on $\chi(\mathscr{O}_X)$ and the appearing fiber types, we restate [12], Proposition 5.1.6:

Proposition 8.2. *Let $f\colon X \to S$ be a relatively minimal genus-1-fibration or quasi-elliptic fibration. Then*

$$c_2(X) = e(X) = e(X_{\overline{\eta}})e(S) + \sum_{s \neq \eta}(e(X_s) - e(X_{\overline{\eta}}) + \delta_s)$$

holds, where $\delta_s \geq 0$. In case that $p \neq 2, 3$ holds or f is quasi-elliptic or that X_s is of type I_n, the equality $\delta_s = 0$ holds. Furthermore, the l-adic Euler characteristic of a fiber $X_s = mF$ is given by

$$e(mF) = \begin{cases} 0 & \textit{if } F \textit{ is of type } I_0 \\ b_2(mF) & \textit{if } F \textit{ is of type } I_n \\ 1 + b_2(mF) & \textit{else.} \end{cases}$$

The second Betti number coincides with the number of irreducible components.

Corollary 8.3. *The Euler characteristic $\chi(\mathscr{O}_X) = 12^{-1}e(X)$ is non-negative.*

Proof. We already showed in the proof of Theorem 1.9 that $(K_X \cdot K_X) = 0$. Now from the preceding theorem, we can read off $e(X) \geq 0$ in the case that X_K is of Kodaira type I_0. If $f\colon X \to S$ is quasi-elliptic, this follows from the following lemma. □

Lemma 8.4. *Let $f\colon X \to S$ be an quasi-elliptic fibration. Then all fibers are of additive type.*

Proof. As the first Betti number b_1 is lower semi-continuous and the generic fiber is of additive type, we have $b_1(X_K) = 0$ and $b_1(X_s) = 0$ for every other fiber. Hence, X_s is of additive type. □

Corollary 8.5. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration. Then $\chi(\mathscr{O}_X) = 0$ if and only if each fiber is of type I_0.*

Proof. As $e(X) = 12\chi(\mathscr{O}_X)$, Proposition 8.2 yields

$$\chi(\mathscr{O}_X) = 0 \cdot e(X) + \sum(e(X_s) - 0 + \delta_s) = \sum(e(X_s) - 0 + \delta_s).$$

Therefore, $\chi(\mathscr{O}_X) = 0$ implies $e(X_s) = \delta_s = 0$, which in turn means that X_s is of type I_0. If one now assumes that all X_s are of type I_0, then $\delta_s = 0$ holds and hence $\chi(\mathscr{O}_X) = 0$. $\square$

We now turn to the Kodaira classification of relatively minimal genus-1-fibrations. Recall that the Kodaira dimension $\kappa(X) \in \{-\infty, 0, 1, 2\}$ is defined as follows: Either $H^0(X, \omega_X^{\otimes t}) = 0$ for all $t > 0$ and we set $\kappa(X) = -\infty$ or we define $\kappa(X)$ to be the minimal non-zero integer n such that $\dim_k H^0(X, \omega_X^{\otimes t}) \cdot t^{-n}$ is bounded for $t \geq 1$. Remember that the canonical bundle formula from Theorem 1.9 says that

$$\omega_X = f^*(\mathscr{L}^\vee \otimes \omega_S) \otimes \mathscr{O}_X\big(\sum_i a_i F_i\big)$$

holds, where $m_i F_i = X_{s_i} = f^*(\{s_i\})$ as Weil divisors. In particular, writing $\mathscr{N} = \mathscr{L}^\vee \otimes \omega_S$ and $x_i = \lfloor t a_i m_i^{-1} \rfloor$, we use the projection formula to see

$$\begin{aligned}
f_*(\omega_X^{\otimes t}) &= f_* f^* \Big(\mathscr{N} \otimes \mathscr{O}_S\Big(\sum_i x_i\{s_i\}\Big)\Big) \otimes \mathscr{O}_X\Big(\sum_i \Big(\frac{t a_i}{m_i} - x_i\Big) m_i F_i\Big) \\
&= \mathscr{N} \otimes \mathscr{O}_S\Big(\sum_i x_i\{s_i\}\Big) \otimes f_*\mathscr{O}_X\Big(\sum_i \underbrace{\Big(\frac{t a_i}{m_i} - x_i\Big) m_i}_{< m_i} F_i\Big) \\
&= \mathscr{N} \otimes \mathscr{O}_S\Big(\sum_i x_i\{s_i\}\Big).
\end{aligned}$$

Note that we used $f_*\mathscr{O}_X(xF_i) = \mathscr{O}_S$ for $0 \leq x \leq m_i - 1$ as proven in Theorem 1.9. In particular, $f_*(\omega_X^{\otimes t})$ is an invertible sheaf, of which we can compute the degree:

$$\deg(f_*(\omega_X^{\otimes t})) = t(2h^1\mathscr{O}_S - 2 + \chi(\mathscr{O}_X) + l(\mathscr{T})) + \sum_i \left\lfloor \frac{t a_i}{m_i} \right\rfloor.$$

To get rid of the floor functions, we take t to be the least common multiple m of the m_i, and define

$$\lambda(f) = \frac{1}{m} \deg(f_*(\omega_X^{\otimes m})) = 2h^1\mathscr{O}_S - 2 + \chi(\mathscr{O}_X) + l(\mathscr{T}) + \sum_i \frac{a_i}{m_i} \in \frac{1}{m}\mathbb{Z}.$$

Note that $H^0(X, \omega_X^{\otimes m}) = H^0(S, f_*(\omega_X^{\otimes m}))$ holds, and the growth of $h^0((f_*\omega_X^{\otimes m})^{\otimes n})$ is determined by its degree. So it is natural to reformulate the Kodaira dimension in terms of $\lambda(f)$. The following proposition is also stated in [52], Proposition 1:

Proposition 8.6. *In above situation, the following holds:*

$$\kappa(X) = \begin{cases} -\infty & \textit{if } \lambda(f) < 0, \\ 0 & \textit{if } \lambda(f) = 0, \\ 1 & \textit{if } \lambda(f) > 0. \end{cases}$$

Proof. If $\lambda(f) < 0$, one finds that $\deg(f_*(\omega_X^{\otimes t})) \leq t\lambda(f) < 0$ for any $t \geq 1$. If $h^0(\omega_X^{\otimes t}) > 0$ holds for some $t \geq 1$, any non-trivial section would give rise to an effective Cartier divisor D. This yields $\deg(f_*(\omega_X)^{\otimes t}) = h^0 \mathscr{O}_D > 0$, a contradiction.

Now assume $\lambda(f) > 0$. Then $f_*(\omega_X^{\otimes m})$ is ample. Consider $\mathscr{F}_i = f_*(\omega_X^{\otimes i})$ for $0 \leq i \leq m-1$ as a mere coherent sheaf on S. Then the global sections of $f_*(\omega_X^{\otimes mn+i}) = f_*(\mathscr{F}_i \otimes \omega_X^{\otimes mn})$ grow linearly in n for $0 \leq i \leq m-1$ (cf. [4], Theorem 1.1, together with [26], Chapter III, Theorem 5.2). In particular, $h^0\omega_X^{\otimes n}$ is not bounded, whereas $n^{-1}h^0\omega_X^{\otimes n}$ is, and $\kappa(X) = 1$.

Now consider the last case $\lambda(f) = 0$. As we have $\chi(\mathscr{O}_X) \geq 0$ by Corollary 8.3, we deduce from the equation $\lambda(f) = 0$ that the genus of S is either zero or one. Assume that S has genus one. Then $l(\mathscr{T}) = 0$ and $a_i = 0$. Using Theorem 1.9, we obtain $\omega_X = f^*\mathscr{L}^\vee$, where $0 = \deg(\mathscr{L}) = \chi(\mathscr{L})$. As we have Kodaira dimension zero, $h^0\omega_X^{\otimes n}$ is positive for some n. Hence $\mathscr{L}^{\otimes -n} = \mathscr{O}_S$ and $\omega_X^{\otimes n} = \mathscr{O}_X$. In particular, the plurigenera are bounded. Now assume that S has genus zero, i.e. $S \simeq \mathbb{P}^1_k$. The only invertible sheaf of degree 0 on $\mathbb{P}^1_k$ is $\mathscr{O}_{\mathbb{P}^1_k}$, therefore $\omega_X^{\otimes m} = \mathscr{O}_X$. □

Example 8.7. We resume Example 1.10 and compute $\lambda(f)$ for the canonical bundle $\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(m-d-2)) \otimes \mathscr{O}_X((p-b-1)F)$. Namely, recalling $m = dp + b$, we have

$$\lambda(f) = -2 + 0 + m - d + \frac{p-b-1}{p} = m - 1 - \frac{m+1}{p}.$$

Therefore, we conclude for the Kodaira dimension

$$\kappa(X) = \begin{cases} -\infty & m = 1, \\ 0 & m = 2 \text{ and } p = 3 \text{ or } m = 3 \text{ and } p = 2, \\ 1 & \text{else.} \end{cases}$$

In general, the torsion in the cohomology is given by

$$\mathscr{T} = (R/\mathfrak{m}^{d+1})^{\oplus b} \oplus (R/\mathfrak{m}^d)^{\oplus p-b-1},$$

cf. Example 2.14. Hence, in the small Kodaira dimensions, $\mathscr{T}$ is given by

$$\mathscr{T} = \begin{cases} R/\mathfrak{m} & m = 1, \\ (R/\mathfrak{m})^{\oplus 2} & m = 2 \text{ and } p = 3, \\ R/\mathfrak{m}^2 & m = 3 \text{ and } p = 2. \end{cases}$$

Example 8.8. We resume Example 2.15 and compute $\lambda(f)$ for the canonical bundle $\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(m-d-2)) \otimes \mathscr{O}_X((n-1)E_0) \otimes \mathscr{O}_X((pn-bn-1)E_\infty)$. Namely, recalling $m = dp + b$, we have

$$\lambda(f) = -2 + 0 + m - d + \frac{n-1}{n} + \frac{pn-bn-1}{pn} = m - \frac{nm+p+1}{np}.$$

Therefore, assuming $n > 1$ as we would otherwise have the previous example, we conclude for the Kodaira dimension $\kappa(X)$ that

$$\kappa(X) = \begin{cases} 0 & (m,n,p) = (1,2,3) \text{ or } (1,2,3), \\ 1 & \text{else.} \end{cases}$$

Recall that a relatively minimal fibration does not admit (-1)-curves in the fibers by definition. Yet, there might exist horizontal (-1)-curves in X, i.e. curves that map surjectively onto S. An (-1)-curve is isomorphic to $\mathbb{P}^1_k$, and restricting X to the divisor gives a finite morphism from $\mathbb{P}^1_k \to S$. Thus, by the Hurwitz Formula, S itself is a projective line. We will call a regular proper surface over k without (-1)-curves a *minimal surface* (in contrast to *minimal fibration*). As we will see, both definitions coincide for Kodaira dimension greater than $-\infty$.

Lemma 8.9. *Let $f\colon X \to S$ be a relatively minimal genus-1-fibration with $\lambda(f) \geq 0$. Then K_X is nef, i.e. $(K_X \cdot C) \geq 0$ for every integral curve $C \subset X$, and X is a minimal surface.*

Proof. If K_X is nef and there would be a (-1)-curve $E \subset X$, then

$$0 \leq (K_X \cdot E) = -1$$

by the very definitions. Contradiction!

So we just have to show that K_X is nef under the assumption $\lambda(f) \geq 0$. Take m to be the least common multiple of the multiplicities of the fibers. Then we saw in the definition of $\lambda(f)$ that mK_X is the pullback of some divisor on S. Let $C \subset X$ be an integral curve. In case C is a vertical curve, i.e. its support lies in some fibers, either $C \cap K_X = \varnothing$ or $C \leq K_X$ as divisors. In both cases, $(K_X \cdot C) = 0$ as we saw in the proof of Theorem 1.9.

Now assume that C is a horizontal divisor. Then C is the closure of some point P closed in the generic fiber X_K and $(X_s \cdot C) = [\kappa(P)\colon K] \geq 0$ holds (cf. [45], Proposition 9.1.30). Hence, writing $mK_S = \sum_i \lambda_i X_{s_i}$ for some $\lambda_i \in \mathbb{Z} \setminus \{0\}$, we deduce

$$\begin{aligned}(mK_X \cdot D) &= \sum_{\lambda_i > 0} \lambda_i (X_{s_i} \cdot C) - \sum_{\lambda_i < 0} |\lambda_i| (X_{s_i} \cdot C) \\ &= \sum_{\lambda_i > 0} \lambda_i [\kappa(P)\colon K] - \sum_{\lambda_i < 0} |\lambda_i| \; [\kappa(P)\colon K] \\ &= \sum_i \lambda_i [\kappa(P)\colon K] \\ &= \deg(f_* m K_X)[\kappa(P)\colon K] \\ &= m\lambda(f)[\kappa(P)\colon K] \geq 0.\end{aligned}$$

□

Proposition 8.10. *Let $f\colon X \to S$ be a relatively minimal genus-1-fibration. Then X is a minimal surface if and only if X is not rational.*

Proof. Assume that X is not rational. By the lemma above, only the case $\lambda(f) < 0$ is unsettled. This means that we can assume $\kappa(X) = -\infty$ and therefore $h^0(\omega_X^{\otimes t}) = 0$ for all $t \geq 1$. Using Corollary 8.3 and Serre duality $h^2 \mathscr{O}_X = h^0 \omega_X = 0$, we have

$$0 \leq \chi(\mathscr{O}_X) = h^0 \mathscr{O}_X - h^1 \mathscr{O}_X + h^2 \mathscr{O}_X = 1 - h^1 \mathscr{O}_X,$$

i.e. $h^1 \mathscr{O}_X \leq 1$. Now Castelnuovo's Rationality Criterion, proven for positive characteristic by Zariski in [76], asserts that $h^1 \mathscr{O}_X = 0$ would imply X rational. Hence, we have to lead $h^1 \mathscr{O}_X = 1$ to a contradiction. Assuming that there is some (-1)-curve $E \subset X$, we can contract it to get a minimal model X'. This in turn must be a ruled surface according to [4], Theorem 13.13. On the one hand, $(K_{X'} \cdot K_{X'}) = 0$ holds due to [4], Proposition 11.19. On the other hand, the intersection number goes down with each blow-up: Let $\pi\colon X \to X'$ be the contraction morphism. Then

$$\begin{aligned}(K_X \cdot K_X) &= \big((\pi^* K_{X'} + E) \cdot (\pi^* K_{X'} + E)\big) \\ &= (\pi^* K_{X'} \cdot \pi^* K_{X'}) + 2(\pi^* K_{X'} \cdot E) + (E \cdot E) \\ &= (\pi^* K_{X'} \cdot \pi^* K_{X'}) - 1 \\ &= -1\end{aligned}$$

in contradiction to $(K_X \cdot K_X) = 0$ as proved in Theorem 1.9.

Now assume that X is a rational surface. Then it is the blow-up of (possibly infinitely near) nine points on $\mathbb{P}^2_k$, cf. [25], Proposition 4.1 and Lemma 4.2 and has (-1)-curves. □

Note that we are not interested in rational surfaces as no wild fibers appear:

Proposition 8.11. *Let $f\colon X \to S$ be a fibered surface such that $h^1\mathscr{O}_X = 0$, e.g. if X rational. Then there are no wild fibers.*

Proof. The first terms of the Leray–Serre spectral sequence

$$0 \longrightarrow H^1(S, \mathscr{O}_S) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^0(S, R^1f_*\mathscr{O}_X) \longrightarrow 0$$

yield $H^0(S, R^1f_*\mathscr{O}_X) = 0$. Therefore, the torsion in $R^1f_*\mathscr{O}_X$ is trivial. □

We now want to understand which wild elliptic fibrations may appear for Kodaira dimension $-\infty$ and zero. As a warm-up, we gather several easy observations. In particular, we will see that no wild quasi-elliptic fibrations appear for $\lambda(f) \leq 0$ except of non-classical supersingular Enriques surfaces:

Proposition 8.12. *Let $f\colon X \to S$ be a relatively minimal genus-1-fibration or quasi-elliptic fibration of Kodaira dimension $\kappa(X) \leq 0$ with at least one wild fiber. Then the following statements hold:*

(i) $S \simeq \mathbb{P}^1_k$.

(ii) $\chi(\mathscr{O}_X) \leq 1$. *If moreover f is quasi-elliptic, then $\chi(\mathscr{O}_X) = 1$.*

(iii) If f is quasi-elliptic, then $p = 2$ and X is a non-classical supersingular Enriques surface, i.e. an Enriques surface with $H^1(X, \mathscr{O}_X) = k$ on which the Frobenius acts as the zero map. It has one wild fiber for which $a = 0$ holds.

(iv) There are at most two (resp. three) multiple fibers if $\lambda(f) < 0$ (resp. $\lambda(f) = 0$).

In particular, there are no quasi-elliptic fibrations with a wild fiber satisfying $\lambda(f) < 0$.

Proof. Recall that

$$\lambda(f) = 2h^1\mathscr{O}_S - 2 + \chi(\mathscr{O}_X) + \underbrace{l(\mathscr{T})}_{\geq 1} + \sum_i \frac{a_i}{m_i}$$

holds. Thus, $\lambda(f) \leq 0$ forces $h^1\mathscr{O}_S = 0$ and $\chi(\mathscr{O}_X) \leq 1$, so in particular $S \simeq \mathbb{P}^1_k$. If f is quasi-elliptic, we apply the Noether Formula, which gives $12 \cdot \chi(\mathscr{O}_X) = e(X)$ using Corollary 8.3. As we know that the generic fiber is a cusp, Proposition 8.2 tells us that

$$e(X) = 2e(S) + \sum (e(X_s) - 2) = 4 + \sum (e(X_s) - 2) > 0$$

holds: We have $e(\mathbb{P}^1_k) = 2$ and $e(X_s) - 2 \geq 0$, the latter one because X_s is of additive type by Lemma 8.4. Therefore, $\chi(\mathscr{O}_X) = 1$.

We still assume that f is quasi-elliptic. Then there is no tame multiple fiber as else some $a_i m_i^{-1} > 0$ and $\chi(\mathscr{O}_X)+l(\mathscr{T})+\sum_i a_i m_i^{-1} > 2$. Similarly, two wild fibers would mean that $l(\mathscr{T}) \geq 2$, leading to $\lambda(f) > 0$, which contradicts our general assumption on $\lambda(f)$. Hence there is only one wild fiber with $a = 0$. As $\lambda(f) = 0$, the surface X is minimal. As $\chi(\mathscr{O}_X) = 1$, it must be an Enriques surface, which is non-classical as $h^1\mathscr{O}_X = 0$ would not allow a wild fiber due to Proposition 8.11. Now from [12], Theorem 5.7.2, or [43], §7.3, we conclude that X must be supersingular to bear a quasi-elliptic structure.

Let us turn to assumption (iv). If $\lambda(f) < 0$, we must have $l(\mathscr{T}) = 1$. Hence there is just one wild fiber. If we would have additionally two tame fibers, then

$$\sum_i \frac{a_i}{m_i} \geq \frac{m_1 - 1}{m_1} + \frac{m_2 - 1}{m_2} > \frac{1}{2} + \frac{1}{2} = 1,$$

which contradicts $\lambda(f) < 0$. The same argument shows that there cannot be three tame fibers for $\lambda(f) = 0$. □

The following lemma will be used in the next theorems.

Lemma 8.13. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration and $l(\mathscr{T}_s) = 1$ for a closed point $s \in S$. Then $m_s = \nu_s p$ and $a_s = \nu_s p - \nu_s - 1$ hold.*

Proof. To ease notation, we omit the index s. Because $l(\mathscr{T}) = 1$, we have $h^1(\mathscr{O}_{mF}) = 2$ for the fiber mF over s by Theorem 2.9. Furthermore, using the notation of Lemma 2.5, the first jumping numbers that increase $h^1(\mathscr{O}_{nF})$ by one are 1 and $1+\nu$. So either the jumping distance grows at $n_1 = 1+\nu$ or $\nu p^e = m < 1 + 2\nu$ holds. The latter one can only be the case for $e = 1$ and $p = 2$. But then $\nu\alpha_0 = 1 + a \leq m = \nu p$ forces $\alpha_0 = 1$, so we have $a = \nu - 1$ and $1 = l(\mathscr{T}) = \alpha_1$ by Corollary 2.8. Hence

$$n_1 + \nu - 1 = n_1 + a = \alpha_1 \nu\, 2 = 2\nu,$$

and therefore $n_1 = \nu + 1$ in all cases.

The next jumping number after $1+\nu$ must then be $1+\nu+p\nu$, and we have the inequality $1 + \nu + \nu p > m = \nu p^e$ from $h^1\mathscr{O}_{mF} = 2$. This is equivalent to $\nu p(p^{e-1} - 1) < 1 + \nu \leq 2\nu$. Hence $e = 1$ and $m = \nu p$. Furthermore, $a < \nu p$ holds and we have

$$\alpha_1 \nu p = n_1 + a < 1 + \nu + \nu p,$$

which is only possible if $\alpha_1 = 1$. Thus $a = p\nu - \nu - 1$. □

Remark 8.14. The lemma shows that *strange fibers*, i.e. wild fibers with $a = m - 1$ (see [31]), do not appear for wild fibers with torsion length one.

For the following theorems, we set up notation: Let $f\colon X \to S$ be a relatively minimal elliptic fibration with canonical bundle formula

$$\omega_X = f^*(\mathscr{L}^\vee \otimes \omega_S) \otimes \mathscr{O}_X\big(\sum_{i=1}^{r} a_i F_i\big)$$

where the first l fibers $m_i F_i$ are wild fibers and the other tame. Following [7], we simply write $(a_1/m_1^*, \ldots, a_l/m_l^*, (m_{l+1}-1)/m_{l+1}, \ldots, (m_r-1)/m_r)$ in shorthand for the multiple fibers appearing in the fibration. Note that the asterisk indicates a wild fiber.

Theorem 8.15. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration of Kodaira dimension $-\infty$ with at least one wild fiber. Then $S \simeq \mathbb{P}^1_k$ and $\chi(\mathscr{O}_X) = 0$. Moreover, all fibers are of type I_0. The possible multiple fibers and the canonical bundle can be read off from the following table:*

multiple fibers	characteristic	canonical bundle
$(p-2/p^*)$	$p \geq 2$	$\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(-1)) \otimes \mathscr{O}_X((p-2)F)$
$(0/2^*, 1/2)$	$p = 2$	$\omega_X = f^*(\mathscr{O}_{\mathbb{P}^1_k}(-1)) \otimes \mathscr{O}_X(F_{\mathrm{tame}})$

Both possibilities appear.

Proof. Proposition 8.11 states that X cannot be rational under the assumption of wildness. Hence, by Proposition 8.10, it is a minimal surface and therefore geometrically ruled. For those, the formula $(K_X \cdot K_X) = 8(1 - h^1\mathscr{O}_X)$ holds, cf. [4], Proposition 11.19. Thus, we conclude $h^1\mathscr{O}_X = 1$ and $\chi(\mathscr{O}_X) = 0$. Applying Corollary 8.5, all singular fibers are of type I_0. The inequality $\lambda(f) < 0$ forces $h^1\mathscr{O}_S = 0$, that is $S \simeq \mathbb{P}^1_k$. Furthermore, in all cases, $\deg(\mathscr{L}) = -\chi(\mathscr{O}_S) - l(\mathscr{T}) = -1$, which leads to the term $f^*(\mathscr{O}_{\mathbb{P}^1_k}(-1))$ in the canonical bundle formula.

Collecting our facts, the formula for $\lambda(f)$ becomes

$$\lambda(f) = -2 + l(\mathscr{T}) + \sum \frac{a_i}{m_i} < 0.$$

It immediately follows that $l(\mathscr{T}) = 1$, and that there is at most (and at least by assumption) one wild fiber F_1. If this is the only multiple fiber, its multiplicity must be a power of p and the order of its normal bundle is one by [31], Corollary 4.2. The wildness automatically follows in this situation. From $l(\mathscr{T}) = 1$, it follows that $m = p$ and $a = p - 1 - 1$ by Lemma 8.13. Such a surface is given in Example 8.7.

Now assume that we have more than one multiple fiber, the wild one indexed by 1. We know from Proposition 8.12 that there is at most one additional multiple fiber m_2F_2, which has to be tame. The inequality $\lambda(f) < 0$ then gives us the condition

$$\frac{a_1}{m_1} + \frac{m_2 - 1}{m_2} = \frac{m_1 - 1 - \nu_1}{m_1} + \frac{m_2 - 1}{m_2} < 1 \quad \Longleftrightarrow \quad 1 < \frac{1 + \nu_1}{m_1} + \frac{1}{m_2},$$

where we again used $m_1 = \nu_1 p$ and $a_1 = \nu_1 p - \nu_1 - 1$ by Lemma 8.13.

We first consider the case $m_1 = m_2$. Then $\nu_1 p = m_1 < 2 + \nu_1$ holds, which is only satisfied if $\nu_1 = 1$ and $p = 2$ hold. This gives us the last case in the table of the statement. Such a surface was constructed in [30], Example 5.2.

Now we lead $m_1 \neq m_2$ to a contradiction: First of all, we notice that the inequality

$$1 < \frac{1}{\nu_1 p} + \frac{1}{p} + \frac{1}{m_2} \leq \frac{2}{p} + \frac{1}{2}$$

holds, from which we deduce $p = 2$ or $p = 3$. By [31], Theorem 3.3, the condition U_2 defined there holds, i.e. there exists an integer l such that

$$\frac{l}{m_1} + \frac{1}{m_2} \in \mathbb{Z}.$$

Reformulating, we have $lm_2 + m_1 \in m_1 m_2 \mathbb{Z}$, from which we see that m_2 divides m_1. Write $m_1 = \gamma m_2$. By assumption, $\gamma \geq 2$.

If $p = 3$, then we estimate $m_2 \geq 2$ and $m_1 \geq 6$ (note that p and m_2 divide m_1). Thus

$$1 < \frac{1}{3} + \frac{1}{m_1} + \frac{1}{m_2} \leq \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1,$$

a contradiction!

If $p = 2$, then by the same estimates as above, we conclude that only $m_2 = 2$ can be possible. For this case, we use again [31], Theorem 3.3, now on condition U_1, which states the existence of integers l_1, l_2 such that

$$\frac{1 + l_1 \nu_1}{m_1} + \frac{l_2}{m_2} \in \mathbb{Z}.$$

In our situation, this reads as

$$\frac{1}{m_1} + \frac{l_1}{2} + \frac{l_2}{2} \in \mathbb{Z},$$

which is impossible as $m_1 \geq 2m_2 = 4$. □

The next theorem gives an overview over all possible relatively minimal elliptic surfaces appearing in Kodaira dimension zero. Whereas the classification of Theorem 8.15 seems to be new, the possible cases in Kodaira dimension zero were already tabulated by Bombieri and Mumford in their paper [7] at the end of §2. They also noted which possibility they could realize. Mitsui studied in [51], §6, the list given by Bombieri and Mumford with regard to the existence of possible types. He gave examples for all occurring possibilities where $\chi(\mathscr{O}_X) = 0$ holds, except for three cases. These are the cases when $l(\mathscr{T}) = 1$ and the fibers are of the form $(1/2^*, 1/2)$, $(1/4^*, 3/4)$ and $(2/4^*, 1/2)$. We narrow the list by excluding the cases $(1/2^*, 1/2)$ and $(2/4^*, 1/2)$:

Theorem 8.16. *Let $f\colon X \to S$ be a relatively minimal elliptic fibration of Kodaira dimension zero.*

(i) If X is a $K3$ surface, then there are no multiple fibers.

(ii) If X is an Enriques surface, the following two cases can occur: Either X is classical and has two tame fibers of multiplicity two or $p = 2$ and X is non-classical with exactly one wild multiple fiber of multiplicity two and $a = 0$.

(iii) If X is an abelian surface, it is given by the quotient of a trivial elliptic fibration $E_1 \times_k E_2 \to E_2$ by a finite subgroup of $E_1 \times_K E_2$, which acts by translation such that the projection $E_1 \times_k E_2 \to E_2$ is equivariant. In particular, there are no multiple fibers.

(iv) If X is a hyperelliptic surface, it is isomorphic to the quotient of two elliptic curves $E_1 \times_k E_2$ by a finite subgroup scheme $A \subset E_2$, which acts on the first factor via an injective homomorphism $\alpha\colon A \to \mathrm{Aut}_{(\mathrm{Grp})}(E_1)$ and on the second factor by translation. This yields two elliptic fibration structures $f_1\colon X \to \mathbb{P}^1_k$ and $f_2\colon X \to E_2/A$, where f_2 has smooth fibers. Thus, each fiber has multiplicity one and is of type I_0. For f_1, only the types which are listed in Table 2 can appear.

Remark 8.17. Bombieri and Mumford as well as Mitsui only listed for $l(\mathscr{T}) = 2$ the possible cases $(0/2^{a*})$ and $(0/3^{b*})$ without specifying the exponent.

Proof. Let us assume that X is a $K3$ surface. Then $\chi(\mathscr{O}_X) = 2$ and hence $\lambda(f) = 0$ if and only if $S \cong \mathbb{P}^1_k$ and there are no multiple fibers.

Now let X be an Enriques surface. Then $\chi(\mathscr{O}_X) = 1$ and the tameness means that $l(\mathscr{T}) = 0$ and $\lambda(f) = 0$ implies $\sum(m_i - 1)m_i^{-1} = 1$, which is only the case for two fibers of multiplicity two. If there is a wild fiber, $l(\mathscr{T}) = 1$ holds and there is exactly one multiple fiber, for which $am^{-1} = 0$ must hold.

Suppose that X is an abelian surface that admits an elliptic fibration. The general fiber is an elliptic curve. Let us translate it in X so that it is a subgroup scheme $E_1 \subset X$. In fact, every closed fiber is a copy of E_1: Take a closed point $x \in X_s$ and consider $x + E_1$. It does not intersect E_1, hence it must be a vertical curve and therefore all of X_s (recall that every closed fiber is an elliptic curve by Corollary 8.5). In particular, all fibers are reduced, the base $S = X/E_1$ is a geometric quotient and therefore an elliptic curve as

$l(\mathcal{T})$	Multiple Fibers	Torsion structure	p	Example or reference
0	$(1/2, 2/3, 5/6)$	0	> 0	[51], §6
	$(1/2, 3/4, 3/4)$	0	> 0	[51], §6
	$(2/3, 2/3, 2/3)$	0	> 0	[51], §6
	$(1/2, 1/2, 1/2, 1/2)$	0	> 0	[51], §6
1	$(0/2^*, 1/2, 1/2)$, $\nu_1 = 1$	$\kappa(s)$	2	[51], §6
	$(1/2^*, 1/2)$, $\nu_1 = 1$	$\kappa(s)$	2	Does not appear
	$(1/3^*, 2/3)$, $\nu_1 = 1$	$\kappa(s)$	3	[51], §6
	$(1/4^*, 3/4)$, $\nu_1 = 2$	$\kappa(s)$	2	Not known
	$(2/4^*, 1/2)$, $\nu_1 = 1$	$\kappa(s)$	2	Does not appear
	$(2/6^*, 2/3)$, $\nu_1 = 3$	$\kappa(s)$	2	Example 8.8
	$(3/6^*, 1/2)$, $\nu_1 = 2$	$\kappa(s)$	3	Example 8.8
2	$(0/2^*)$, $\nu_1 = 1$	$R/\mathfrak{m}^2$	2	Example 8.7
	$(0/4^*)$, $\nu_1 = 1$	$\kappa(s) \oplus \kappa(s)$	2	[31], §8.1 and §8.2
	$(0/3^*)$, $\nu_1 = 1$	$\kappa(s) \oplus \kappa(s)$	3	Example 8.7
	$(0/2^*, 0/2^*)$, $\nu_i = 1$	$\kappa(s) \oplus \kappa(s')$	2	[7], §3

Table 2: Potential types of multiple fibers in hyperelliptic surfaces

well. Now secondly, observe that X is isogenous to the product $E_1 \times_k E_2$ for an elliptic curve E_2 over k by Poincaré's Complete Reducibility Theorem (cf. [56], §19, Theorem 1). Consider the commutative diagram

$$\begin{array}{ccc} E_1 \times_k E_2 & \longrightarrow & X \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & S, \end{array}$$

where the left vertical arrow is the second projection and $E_2 \to S$ is given by restricting $E_1 \times_k E_2 \to X \to S$ to E_2. Then both vertical arrows are minimal elliptic fibrations and S is the quotient of E_2 by the kernel of $E_1 \times_k E_2 \to X$ restricted to E_2. Therefore, taking the quotients induces the elliptic fibration $X \to S$.

The case of hyperelliptic surfaces is pretty much the same as for abelian surfaces equipped with an elliptic fibration: It is the quotient of a product of elliptic curves. But this time, the group only acts freely on one component. For a proof, see [7], Proposition 5 and Theorem 4. Thus, we only have to focus on the case where $S = \mathbb{P}^1_k$. Note that the table is already in [7] or [51], §6, but without proof. For convenience, we give one:

Recall that $\lambda(f) = -2 + l(\mathscr{T}) + \sum a_i m_i^{-1}$ holds in our situation and assume $l(\mathscr{T}) = 0$. Then $\lambda(f) = 0$ forces $\sum_{i=1}^{r} (m_i - 1) m_i^{-1} = 2$, where r is the number of multiple fibers.

Reformulating

$$2 = r - \underbrace{\sum_{i=1}^{r} \frac{1}{m_i}}_{\leq r/2} \geq \frac{r}{2},$$

we deduce $r \leq 4$. If $r = 4$, then $m_i = 2$ must hold for $1 \leq i \leq 4$. So we are left to study the case $r \leq 3$. If $m_1 \geq 7$, we deduce from the inequality

$$1 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \leq \frac{1}{7} + \frac{1}{m_2} + \frac{1}{m_3}$$

the inequality $m_2^{-1} + m_3^{-1} \geq 6/7$. This is only satisfied if $m_2 = m_3 = 2$, but then the equality $1 = m_1^{-1} + m_2^{-1} + m_3^{-1}$ is not true, contradiction. Hence, $m_i \leq 6$ must hold. From this point on, it is easy to calculate which combinations satisfy $\lambda(f) = 0$.

Now let $l(\mathscr{T}) = 1$. As every tame fiber contributes at least 2^{-1} to $\lambda(f)$, we have at most two tame fibers. As $(m_1 - 1)m_1^{-1} + (m_2 - 1)m_2^{-1} \geq 1$ holds with equality only if $m_1 = m_2 = 2$, one immediately sees that the constellation $(0/2^*, 1/2, 1/2)$ is the only possible one. On the other hand, we must have at least one multiple tame fiber due to $a_1 m_1^{-1} < 1$. It suffices to check that $m_1 \leq 7$: Then we reformulate the equality $a_1 m_1^{-1} + (m_2 - 1)m_2^{-1} = 1$ to $m_2 a = m_1$ and there are only finitely many cases left to study. We already deduced $m_1 = \nu p$ and $a_1 = m_1 - 1 - \nu_1$ in Lemma 8.13. Hence, if $m_1 \geq 7$, we reformulate $m_2 a = m_1$ to obtain the inequality

$$\frac{1}{m_2} = \frac{a_1}{m_2} = 1 \underbrace{- \frac{1}{m_1}}_{\geq -7^{-1}} \underbrace{- \frac{1}{p}}_{\geq -2^{-1}} \geq 1 - \frac{1}{7} - \frac{1}{2} = \frac{5}{14},$$

which only holds if $m_2 = 2$. In this case, the prime p in above inequality must be two, as otherwise the inequality would not be true. Thus, $m_1 = \nu_1 \cdot 2$ and $m_1 = m_2 a = 2(\nu_1 - 1)$ hold. Contradiction. Hence $m_1 \leq 6$. Now one checks the finitely many possible cases. Note that ν_1 can be determined by the properties $\nu_1 p = m_1$ and ν_1 divides $a_1 + 1$ as well as m_1.

For the remaining case $l(\mathscr{T}) = 2$, one immediately notices that there are at most two wild fibers. The condition $\lambda(f) = 0$ now implies that all $a_i = 0$, in particular there is no tame multiple fiber. If there are two wild fibers, Lemma 8.13 shows that they must be of type $(p-2/p^*)$, hence of type $(0/2^*)$ due to $a = 0$. This settles the last entry in the list and the torsion structure. So assume that there is just one wild fiber. Using the conclusions from Lemma 2.5, we obtain $1 = 1 + a = \alpha_0 \nu$, which tells us that $\alpha_0 = \nu = 1$ holds. The formula for the torsion length $2 = l(\mathscr{T}) = 1 - \alpha_0 + (p-1)\sum_{i=0}^{e} \alpha_i$ from Corollary 2.8 reduces the possible primes to 2 and 3. If $p = 3$, then $\alpha_1 = 1$ and $e = 1$, settling the case $(0/3^*)$. If $p = 2$, either $e = 1$ and $\alpha_1 = 2$ or $e = 2$ and $\alpha_1 = \alpha_2 = 1$, settling the cases

(0/2*) and (0/4*).

For the torsion structure, we apply Theorem 2.11. As $a = 0$ in all cases, the formula for the y_i and z_i becomes easier:

$$y_j = (p-1)\big(\alpha_j - \beta_j p^{e-j}\big),$$
$$z_j = (p-1)\big((\beta_j + 1)p^{e-j} - \alpha_j\big).$$

Furthermore, in all cases, $1 = 1 + a = \nu\alpha_0$ holds, i.e. $\alpha_0 = 1$ and $l(\mathscr{T}) = (p-1)\sum_{i=1}^{e} \alpha_i$. Start with the case (0/4*): We have $e = 2$ due to $\nu = 1$. Therefore, $2 = l(\mathscr{T}) = \alpha_1 + \alpha_2$. Because $p\alpha_j - \alpha_j = k_j > 0$ for $0 \leq j \leq e-1$, we have $\alpha_1, \alpha_2 > 0$, so both are 1 and we compute $n_1 = 2$, $n_2 = 4$, $\beta_1 = 0$, $\beta_2 = 1$ and the numbers $y_1 = 1$, $z_1 = 2$, $y_2 = 0$, $z_2 = 1$. This gives $\mathscr{T} = (R/\mathfrak{m})^{\oplus 2}$.

The other cases are a bit simpler due to Remark 2.13: For (0/2*), we have $e = 1$ and $2 = l(\mathscr{T}) = \alpha_1$. Therefore, $\mathscr{T}$ is isomorphic to $R/\mathfrak{m}^2$. In the case (0/3*), the equality $2 = l(\mathscr{T})$ gives $\alpha_1 = 1$, hence $\mathscr{T} = (R/\mathfrak{m})^{\oplus 2}$. If we have two wild fibers with torsion length 1, we already saw that the torsion structure is $R/\mathfrak{m}$.

Now turning back to $l(\mathscr{T}) = 1$, we can exclude the cases $(1/2^*, 1/2)$ and $(2/4^*, 1/2)$ by the following argument: By Lemma 8.13, a wild fiber with torsion length 1 satisfies $a = m - \nu - 1$. This excludes the appearance of a wild fiber (1/2*). Furthermore, we deduced $e = 1$, which excludes the case (2/4*): because $\nu = 1$, the multiplicity is given by $m = 1 \cdot 2^2$, i.e. $e = 2$. Contradiction. □

Remark 8.18. Proposition 8.1 does not help in constructing an hyperelliptic surface with multiple fibers $(1/4^*, 3/4)$ as in Table 2. If $\kappa(X) \leq 0$ and $f\colon X \to S$ admits a wild fiber, we always deduced that $S \simeq \mathbb{P}^1_k$. According to [12], Corollary 5.5.5, every elliptic fibration $E \to \mathbb{P}^1_k$ with section without degenerate fiber is trivial, i.e. $X \simeq E_k \times_k \mathbb{P}^1_k$ for an elliptic curve E_k over k. Let X_K be a torsor under the elliptic curve E_K. Then the fibers of their relatively minimal proper regular models E and X over S have the same Kodaira type, see Theorem 1.13. That means that E does not admit a degenerate fiber, hence is trivial and we cannot apply Proposition 8.1.

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